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PAPER

Wigner's Semicircle Law of Weighted Random Networks

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SUMMARY Spectral graph theory provides an algebraic approach to investigate the characteristics of weighted networks using the eigenvalues and eigenvectors of a matrix (e.g., normalized Laplacian matrix) that represents the structure of the network. However, it is difficult to accurately represent the structures of large-scale and complex networks (e.g., social network) as a matrix. This difficulty can be avoided if there is a universality, such that the eigenvalues are independent of the detailed structure in large-scale and complex network. In this paper, we clarify Wigner's Semicircle Law for weighted networks as such a universality. The law indicates that the eigenvalues of the normalized Laplacian matrix of weighted networks can be calculated from a few network statistics (the average degree, average link weight, and square average link weight) when the weighted networks satisfy a sufficient condition of the node degrees and the link weights.

key words: *random matrix theory, Wigner's semicircle law, spectral graph theory, Laplacian matrix, network analysis*

1. Introduction

Many networks such as railway networks, social networks, the Internet, and airport networks are modeled as weighted networks that are composed of nodes and weighted links [1]–[3]. The weighting of links is important in network modeling, but it is a tedious task in the case of large-scale and complex networks, such as social networks. In a social network, nodes and links correspond to persons and their relationships, respectively. To assign the correct weight to each link in the social network, the strength of the relationships among people should be accurately estimated from a huge amount of personal data (e.g., communication histories in mobile phones and social media). Owing to the risk of breaching privacy and the associated computational complexity, it is unrealistic to gather such personal data, and calculate the strength of the relationships accurately.

Spectral graph theory provides an algebraic approach to investigate the characteristics of weighted networks using the eigenvalues and the eigenvectors of a matrix (e.g., normalized Laplacian matrix) that represents the structure of the network [4], [5]. In particular, the eigenvalues are important to understand the characteristics related to the entire network on the basis of spectral graph theory. In [6], it has been demonstrated that the eigenvalue distribution of the normalized Laplacian matrix affects the information dissemination speed throughout the social network. In general, the eigenvalues of the matrix are calculated from all the matrix elements. The elements of the matrix representing the structure of a weighted network are determined by the weights on the links. Hence, the weighting of the links is required to apply spectral graph theory for weighted networks. This would deter the application of spectral graph theory for large-scale and complex networks. However, the weighting of the links is avoidable if there is a universality that the eigenvalues are independent of the detailed structure (e.g., the weight of each link) of large-scale and complex networks. Therefore, finding of such a universality of the eigenvalues expands the applicable region of spectral graph theory.

Random matrix theory discusses a universality of the eigenvalues if the elements of the matrix are given by random variables [7]–[9]. In [9], Chung et al. analyzed the random matrix corresponding to the normalized Laplacian matrices of unweighted networks, and have confirmed the universality (Wigner's semicircle law) that the eigenvalues of the normalized Laplacian matrix follow the semicircle distribution. In [6], it was investigated whether Wigner's semicircle law [9] of unweighted networks is satisfied for weighted networks, through numerical experiments. However, the condition of weighted networks satisfying Wigner's semicircle law was not explicitly furnished, and the width of the eigenvalue distribution (spectral radius) was not found. Hence, the universality for weighted networks is not fully understood. Given the available literature, no study has elucidated the universality of the eigenvalues for weighted networks.

In this paper, the universality (Wigner's semicircle law) of the eigenvalues is presented for weighted networks on the basis of the discussion of [9]. Our findings include (a) a sufficient condition for weighted networks to satisfy Wigner's semicircle law and (b) an expression of the spectral radius of the eigenvalue distribution. According to the derived expression, the spectral radius determines the eigenvalue distribution of the normalized Laplacian matrix of weighted networks, and is calculated from a few network statistics (the average degree, average link weight, and square average link weight). Hence, the eigenvalues can be obtained from a few network statistics when the weighted networks satisfy the sufficient condition of the node degrees and the link weights. Using some numerical examples, the validity of the sufficient condition is confirmed.

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This paper is organized as follows. In Sect. 2, spectral graph theory and the random matrix theory for weighted networks is described. In Sect. 3, Wigner's semicircle law for weighted networks is proved on the basis of the random matrix theory. Section 4 shows some numerical examples. Finally, in Sect. 6, we conclude this paper and discuss the future work.

2. Preliminary

2.1 Spectral Graph Theory

In spectral graph theory, the structure of a network is represented by a matrix, and the characteristics of the network are investigated using its eigenvalues and eigenvectors. In this section, we describe spectral graph theory for weighted networks.

We denote a weighted network by $G = (V, E, \boldsymbol{w})$ where V and E are the sets of nodes and links, respectively. Let n be the number of nodes in G. The link between nodes i and j is denoted by (i, j). Link (i, j) has the weight $w(i, j)$ where $w(i, j) = w(j, i)$ and $w(i, j) > 0$. Let ∂i be the set of adjacent nodes of node i. The weighted degree d_i of nodes i is defined by

$$d_i := \sum_{j\in\partial i} w(i, j). \tag{1}$$

To represent the structure of links and nodes in G, there are adjacency and degree matrices, $\boldsymbol{A}$ and $\boldsymbol{D}$, respectively. The (i, j)-th element $A(i, j)$ of adjacency matrix $\boldsymbol{A}$ is defined by

$$A(i, j) := \begin{cases} w(i, j) & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Degree matrix $\boldsymbol{D}$ is defined by

$$\boldsymbol{D} := \mathrm{diag}(d_i)_{1\le i\le n}. \tag{3}$$

To represent both the structure of nodes and links in G, a normalized Laplacian matrix $\boldsymbol{N}$ is often used. The normalized Laplacian matrix $\boldsymbol{N}$ is defined by

$$\boldsymbol{N} := \boldsymbol{I} - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}, \tag{4}$$

where $\boldsymbol{I}$ is the identity matrix.

Since a normalized Laplacian matrix $\boldsymbol{N}$ is symmetric ($\boldsymbol{N} = \boldsymbol{N}^\top$), its eigenvalues λ_l ($l = 1, \ldots, n$) are real numbers. Let $\boldsymbol{q}_l$ be the eigenvector of eigenvalue λ_l where ${\boldsymbol{q}_l}^\top \boldsymbol{q}_l = 1$. We assign a number to λ_l in ascending order, and hence λ_l refers to the l-th minimum eigenvalue of $\boldsymbol{N}$. The range of the eigenvalues is given by

$$0 = \lambda_1 < \lambda_2 \le \ldots \le \lambda_n < 2. \tag{5}$$

If $\lambda_2 > 0$, the weighted network G is connected (i.e., there is at least one path between every pair of nodes). The weighted network G is not a bipartite graph, if $\lambda_2 < 2$. We define the spectral radius r by $r := \max_{2\le l\le n} |1 - \lambda_l|$. Spectral radius r satisfies $0 < r < 1$ because $0 < \lambda_l < 2$ for $2 \le l \le n$. Eigenvector $\boldsymbol{q}_1$ of minimum eigenvalue λ_1 is given by

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{\mathrm{Vol}(G)}}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})^\top, \tag{6}$$

where $\mathrm{Vol}(G)$ is defined by

$$\mathrm{Vol}(G) := \sum_{i\in G} d_i. \tag{7}$$

Since the eigenvectors $\boldsymbol{q}_l$ is an orthonormal basis, matrix $\boldsymbol{Q} = (\boldsymbol{q}_l)_{1\le k\le n}$ is the orthogonal matrix ($\boldsymbol{Q}^{-1} = \boldsymbol{Q}^\top$).

Using $\boldsymbol{Q}$ and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_l)_{1\le k\le n}$, the normalized Laplacian matrix $\boldsymbol{N}$ is given by

$$\boldsymbol{N} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^\top = \sum_{l=1}^{n} \lambda_l\, \boldsymbol{q}_l\, \boldsymbol{q}_l^\top. \tag{8}$$

From the above equation, $\boldsymbol{N}$ is determined by eigenvalues $(\lambda_l)_{1\le l\le n}$ and eigenvectors $(\boldsymbol{q}_l)_{1\le l\le n}$. Hence, the weighted network G can be analyzed not only with $\boldsymbol{N}$ but also with its eigenvalues and eigenvectors. Spectral graph theory provides an algebraic method for analyzing G with eigenvalues λ_l and eigenvectors $\boldsymbol{q}_l$ of $\boldsymbol{N}$. In particular, the eigenvalues λ_l are important to understand the statistical characteristics related to the entirety of the weighted network G. In order to calculate eigenvalues λ_l, in general, all elements $N(i, j)$ must be given accurately. However, if there is a useful universality of eigenvalues λ_l, we can investigate the statistical characteristics of the weighted network G without all elements $N(i, j)$.

2.2 Random Matrix Theory

Random matrix theory focuses on random matrices whose the elements are given by random variables, and clarifies that a universality of the eigenvalues appears if the matrix size approaches infinity. When the links and their weights in the weighted network G are randomly given with a stochastic rule, elements $N(i, j)$ of the normalized Laplacian matrix $\boldsymbol{N}$ become random variables, and $\boldsymbol{N}$ can be treated as a random matrix. Note that the universality of eigenvalues λ_l of $\boldsymbol{N}$ corresponds to a characteristic of the statistical ensemble of the random networks generated with the same stochastic rule. Hence, clarifying such a universality contributes to the growth of the statistical mechanics on networks.

In [9], the link between nodes i and j in an unweighted network is randomly generated using the stochastic rule with random variable L_{ij}. If $L_{ij} = 0$, there is no link between nodes i and j. On the other hand, if $L_{ij} = 1$, a link exists between nodes i and j. We denote probability $\mathbb{P}\left[L_{ij} = 1\right]$ by p_{ij}, which is given by

$$p_{ij} = \rho\, \sigma_i\, \sigma_j, \tag{9}$$

where $\sigma_i > 0$ and $\rho = 1/\sum_{i\in V} \sigma_i$. Using Eq. (9), the expectation of the node i's degree is given by σ_i. Let σ_{avg},

$\sigma_{\min}$, and $\sigma_{\max}$ be the average degree, the minimum degree, and the maximum degree. These are defined by $\sigma_{\mathrm{avg}} := 1/n \sum_{i \in V} \sigma_i$, $\sigma_{\min} := \inf_{i \in V} \sigma_i$, and $\sigma_{\max} := \sup_{i \in V} \sigma_i$, respectively. We can write ρ as $\rho = 1/(n\,\sigma_{\mathrm{avg}})$. Similar to [9], it is assumed that $\sigma_{\max}^2 < 1/\rho$, so that $p_{ij} \leq 1$. Using the above stochastic rule, special networks (e.g., networks with no links) are rarely generated. The probability of generating such special networks is very small, and therefore, this stochastic rule poses no problem in the discussion of a universality appearing when $n \to \infty$.

In the weighted network G, the weight of the link (i, j) is randomly set using the stochastic rule with random variable W. In the stochastic rule, the random variable W follows the conditional probability density function $f_{W|L_{ij}}(w \mid l)$, which is defined by

$$f_{W|L_{ij}}(w \mid l)\,\mathrm{d}w := \mathbb{P}\left[w \leq W \leq w + \mathrm{d}w \mid L_{ij} = l\right]. \tag{10}$$

If $L_{ij} = 0$ (i.e., link (i, j) does not exist), W is always 0, and hence $f_{W|L_{ij}}(w \mid 0) = \delta(w)$ where $\delta(x)$ is the Dirac delta function. On the contrary, if $L_{ij} = 1$, then $W > 0$. Since link weights in actual networks cannot be infinity, it is assumed that W is bounded. Using a finite value $w_{\max}$, $W \leq w_{\max}$. For convenience, we write

$$p_W(w) := f_{W|L_{ij}}(w \mid 1). \tag{11}$$

Let $\mathbb{E}_{p_W}[W^m]$ be the m-th moment of W with the condition $L_{ij} = 1$. $\mathbb{E}_{p_W}[W^m]$ is defined by

$$\mathbb{E}_{p_W}[W^m] := \int_0^{w_{\max}} w^m\, p_W(w)\,\mathrm{d}w. \tag{12}$$

Even if the weights of all links in G are divided by $w_{\max}$, the normalized Laplacian matrix $\boldsymbol{N}$ remains invariant. Hence, without loss of generality, we assume $w_{\max} = 1$. With this assumption, $\mathbb{E}_{p_W}[W^m]$ has the following properties: (a) $\mathbb{E}_{p_W}[W^l] \leq \mathbb{E}_{f_W}[W^m]$ for $l > m$, and (b) $\mathbb{E}_{p_W}[W^m] \leq 1$ since $\mathbb{E}_{p_W}[W^0] = 1$.

Following the above stochastic rules, not only elements $N(i, j)$ but also the eigenvalues λ_l $(l = 2, \ldots, n)$ of the normalized Laplacian matrix $\boldsymbol{N}$ for the weighted network G become random variables depending on the set of random variables $\boldsymbol{\Gamma} = (\boldsymbol{L}, W)$ where $\boldsymbol{L} = (L_{ij})_{(i,j) \in V^2}$. Let Λ be the random variable for eigenvalue λ of $\boldsymbol{N}$. We denote the conditional eigenvalue density of eigenvalues λ_l $(l = 2, \ldots, n)$ of $\boldsymbol{N}$ by $f^{(n)}_{\Lambda|\boldsymbol{\Gamma}}(\lambda \mid \gamma)$, which is defined by

$$f^{(n)}_{\Lambda|\boldsymbol{\Gamma}}(\lambda \mid \boldsymbol{\gamma}) := \frac{1}{n-1} \sum_{l=2}^{n} \delta(\lambda - \lambda_l), \tag{13}$$

where λ_l is the function of $\boldsymbol{\gamma}$. Since eigenvalues λ_l vary stochastically, conditional eigenvalue density $f^{(n)}_{\Lambda|\boldsymbol{\Gamma}}(\lambda \mid \boldsymbol{\gamma})$ is also a random variable. Note that $f^{(n)}_{\Lambda|\boldsymbol{\Gamma}}(\lambda \mid \boldsymbol{\gamma})$ can be treated as a conditional probability density function because $\int_{\lambda_2}^{\lambda_n} f^{(n)}_{\Lambda|\boldsymbol{\Gamma}}(\lambda \mid \boldsymbol{\gamma})\,\mathrm{d}\lambda = 1$. Using probability $\mathbb{P}[\boldsymbol{\Gamma} = \boldsymbol{\gamma}]$, eigenvalue density $f^{(n)}_{\Lambda}(\lambda)$ of $\boldsymbol{N}$ is given by

$$\begin{aligned} f^{(n)}_{\Lambda}(\lambda) &= \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[f^{(n)}_{\Lambda|\boldsymbol{\Gamma}}(\lambda \mid \boldsymbol{\gamma})] \\ &= \frac{1}{n-1} \sum_{\boldsymbol{\gamma}} \mathbb{P}\left[\boldsymbol{\Gamma} = \boldsymbol{\gamma}\right] \sum_{l=2}^{n} \delta(\lambda - \lambda_l). \end{aligned} \tag{14}$$

Since $\int_{\lambda_2}^{\lambda_n} f^{(n)}_{\Lambda}(\lambda)\,\mathrm{d}\lambda = 1$, $f^{(n)}_{\Lambda}(\lambda)$ can be also treated as a probability density function. Then, we denote the m-th moment for $1 - \Lambda$ using $f^{(n)}_{\Lambda}(\lambda)$ by $\mathbb{E}_{f^{(n)}_{\Lambda}}[(1 - \Lambda)^m]$, which is defined by

$$\begin{aligned} \mathbb{E}_{f^{(n)}_{\Lambda}}[(1 - \Lambda)^m] &:= \int_{\lambda_2}^{\lambda_n} (1 - \lambda)^m f^{(n)}_{\Lambda}(\lambda)\,\mathrm{d}\lambda \\ &= \frac{1}{n-1} \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}\left[\sum_{l=2}^{n} (1 - \lambda_l)^m\right]. \end{aligned} \tag{15}$$

Using the approach of the random matrix theory, the previous work [9] has established the universality (Wigner's semicircle law) that the eigenvalue density $f^{(n)}_{\Lambda}(\lambda)$ for unweighted networks follows a certain distribution (i.e., semicircle distribution). If such a universality exists even for the weighted network G, eigenvalue density $f^{(n)}_{\Lambda}(\lambda)$ can be obtained without giving link weights $w(i, j)$ correctly. This allows the analysis of G based on spectral graph theory even if G is a large-scale and complex network, such as social network.

3. Wigner's Semicircle Law of Weighted Network *G*

In this section, we probe Wigner's semicircle law for a weighted network G on the basis of the discussion in [9]. Wigner's semicircle law for G is as follows:

Theorem. *If the weighted network G satisfies the degree condition*

$$\sigma_{\min}^2 \gg \frac{\sigma_{\mathrm{avg}}}{\mathbb{E}_{p_W}[W^2]}, \tag{16}$$

$f^{(n)}_{\Lambda}(\lambda)$ of the normalized Laplacian matrix $\boldsymbol{N}$ converges to a semicircle distribution $\tilde{f}_{\Lambda}(\lambda)$ as $n \to \infty$. A semicircle distribution $\tilde{f}_{\Lambda}(\lambda)$ is given by

$$\tilde{f}_{\Lambda}(\lambda) = \begin{cases} \dfrac{2}{\pi\,\tilde{r}^2} \sqrt{\tilde{r}^2 - (1 - \lambda)^2} & 1 - \tilde{r} < \lambda < 1 + \tilde{r} \\ 0 & \text{otherwise} \end{cases}, \tag{17}$$

where $\tilde{r}$ is the limit value of spectral radius r as $n \to \infty$, and is given by

$$\tilde{r} = \frac{2}{\sqrt{\sigma_{\mathrm{avg}}}} \frac{\sqrt{\mathbb{E}_{p_W}[W^2]}}{\mathbb{E}_{p_W}[W]}. \tag{18}$$

Remark. *When the links in G are not weighted, W is always 1 if $L_{ij} = 1$, and hence $\mathbb{E}_{p_W}[W] = \mathbb{E}_{p_W}[W^2] = 1$. By* *substituting them into Eqs.* (16) *and* (18)*, we obtain Wigner's semicircle law for unweighted networks shown in [9]. Therefore, it can be generalized to a weighted network G.*

The degree condition (16) *means that* $\sigma_{\min}$ *is sufficiently large in a weighted network G, and is necessary to satisfy Eqs. (45) and (58) in the proof. How easily Eq.* (16) *holds, depends on the value of* $\sigma_{\min}$*. If* $\sigma_{\min}$ *is large, it is easy to satisfy Eq.* (16)*.*

According to the degree distribution of the Facebook friend network reported in [10], the minimum and the average degrees of Facebook users are 1 *and* 190*, respectively. Since* $\mathbb{E}_{p_W}[W^2]$ *is smaller than* 1*, Eq.* (16) *is not satisfied for the friend network of Facebook users. Note that the survey in [10] targeted all Facebook users who logged in May 2011, and includes not only frequent Facebook users but also infrequent ones. According to [11], frequent users of Facebook tend to have many friends. Hence, the induced subgraph of the frequent Facebook users would have a minimum degree large enough to satisfy Eq.* (16)*. Since the frequent users mainly disseminate information in social networks, the analysis of such an induced subgraph is important to understand the characteristics of the information dissemination. Analysis of the induced subgraph of frequent users in actual SNSs will be done as part of a future study. In this analysis, we obtain the degrees of SNS users with the node sampling method [12], and use the obtained degree as* σ_i*.*

Equation (17) *implies that the width of the eigenvalue distribution of the normalized Laplacian matrix* $\boldsymbol{N}$ *becomes large as* $\tilde{r}$ *increases. From Eq.* (18)*,* $\tilde{r}$ *is determined by the network statistics (*σ_{avg}*,* $\mathbb{E}_{p_W}[W^2]$*, and* $\mathbb{E}_{p_W}[W]^2$*). In particular, if relative variance* $\mathbb{E}_{p_W}[W^2]/\mathbb{E}_{p_W}[W]^2$ *increases,* $\tilde{r}$ *increases. Hence, there is the relationship between the width of the eigenvalue distribution and relative variance* $\mathbb{E}_{p_W}[W^2]/\mathbb{E}_{p_W}[W]^2$*. According to the analysis in [7],* $\tilde{r}$ *of the semicircle distribution increases, as the variance of the non-diagonal elements of the matrix considering Wigner's semicircle law increases. As relative variance* $\mathbb{E}_{p_W}[W^2]/\mathbb{E}_{p_W}[W]^2$ *increases, the variance of the non-diagonal elements of the normalized Laplacian matrix* $\boldsymbol{N}$ *also increases. Thus, the relationship between the width of the eigenvalue distribution and relative variance* $\mathbb{E}_{p_W}[W^2]/\mathbb{E}_{p_W}[W]^2$ *expressed in Eqs.* (17) *and* (18) *is derived.*

Proof. If eigenvalue density $f_\Lambda^{(n)}(\lambda)$ is given by a semicircle distribution $\tilde{f}_\Lambda(\lambda)$, even moment $\mathbb{E}_{\tilde{f}_\Lambda}[(1-\Lambda)^{2m}]$ and odd moment $\mathbb{E}_{\tilde{f}_\Lambda}[(1-\Lambda)^{2m+1}]$ for $1-\Lambda$ are given by

$$\mathbb{E}_{\tilde{f}_\Lambda}[(1-\Lambda)^{2m}] = \int_{1-\tilde{r}}^{1+\tilde{r}} (1-\lambda)^{2m} \tilde{f}_\Lambda(\lambda)\,\mathrm{d}\lambda = \left(\frac{\tilde{r}}{2}\right)^{2m} \frac{(2\,m)!}{m!\,(m+1)!}, \tag{19}$$

$$\mathbb{E}_{\tilde{f}_\Lambda}[(1-\Lambda)^{2m+1}] = \int_{1-\tilde{r}}^{1+\tilde{r}} (1-\lambda)^{2m+1} \tilde{f}_\Lambda(\lambda)\,\mathrm{d}\lambda = 0. \tag{20}$$

Since $\mathbb{E}_{\tilde{f}_\Lambda}[\Lambda] = 1$, the above moments for $1-\Lambda$ correspond to the central moments of Λ. The following are equivalent:

1. As $n \to \infty$, eigenvalue density $f_\Lambda^{(n)}(\lambda)$ converges to semicircle distribution $\tilde{f}_\Lambda(\lambda)$.
2. As $n \to \infty$, even moment $\mathbb{E}_{\tilde{f}_\Lambda}[(1-\Lambda)^{2m}]$ and odd moment $\mathbb{E}_{\tilde{f}_\Lambda}[(1-\Lambda)^{2m+1}]$ converge to Eqs. (19) and (20), respectively.

To prove Wigner's semicircle law for a weighted network G, we show that 2. is fulfilled if the degree condition (16) is satisfied.

For convenience, we use matrix $\boldsymbol{M}$ after removing the effect of minimum eigenvalue λ_1 from normalized Laplacian matrix $\boldsymbol{N}$. Matrix $\boldsymbol{M}$ is defined by

$$\begin{aligned}\boldsymbol{M} &:= \sum_{l=2}^{n} (1-\lambda_l)\,\boldsymbol{q}_l\,\boldsymbol{q}_l^\top = \boldsymbol{I} - \boldsymbol{N} - \boldsymbol{q}_1\,\boldsymbol{q}_1^\top \\ &= \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2} - \frac{1}{\mathrm{Vol}(G)}\boldsymbol{D}^{1/2}\boldsymbol{K}\boldsymbol{D}^{1/2},\end{aligned} \tag{21}$$

where $\boldsymbol{K}$ is the matrix with all elements given by 1, and corresponds to the adjacency matrix for the complete graph including self-loops at all nodes. Matrix $\boldsymbol{M}$ has $n-1$ nonzero eigenvalues, and the l-th largest eigenvalue is given by $1-\lambda_l$. Hence

$$\mathrm{Tr}[\boldsymbol{M}^m] = \sum_{l=2}^{n} (1-\lambda_l)^m. \tag{22}$$

By substituting the above equation into Eq. (15), we obtain

$$\mathbb{E}_{f_\Lambda^{(n)}}[(1-\Lambda)^m] = \frac{1}{n-1}\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{M}^m]]. \tag{23}$$

Since $\mathbb{E}_{p_W}[W]$ is finite, the weighted degree d_i of the node i converges to its expected value $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[d_i]$ as $n \to \infty$. $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[d_i]$ is given by

$$\begin{aligned}\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[d_i] &= \sum_{j\in V} \Big(p_{ij}\mathbb{E}_{f_{W|L_{ij}}}[W \mid L_{ij}=1] \\ &\qquad + (1-p_{ij})\mathbb{E}_{f_{W|L_{ij}}}[W \mid L_{ij}=0]\Big) \\ &= \sum_{j\in V} p_{ij}\mathbb{E}_{p_W}[W] \\ &= \mathbb{E}_{p_W}[W] \sum_{j\in V} \rho\,\sigma_i\,\sigma_j \\ &= \mathbb{E}_{p_W}[W]\sigma_i.\end{aligned} \tag{24}$$

Note that $p_{ij} = \rho\,\sigma_i\sigma_j$ was used to derive the above equation. By substituting $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[d_i]$ into Eq. (21), we obtain matrix $\boldsymbol{C}$, which is given by

$$\boldsymbol{C} = \frac{1}{\mathbb{E}_{p_W}[W]}\boldsymbol{\sigma}^{-1/2}\boldsymbol{A}\boldsymbol{\sigma}^{-1/2} - \rho\,\boldsymbol{\sigma}^{1/2}\boldsymbol{K}\boldsymbol{\sigma}^{1/2}, \tag{25}$$

where $\boldsymbol{\sigma} = \mathrm{diag}(\sigma_i)_{1\le i\le n}$. As $n \to \infty$, matrix $\boldsymbol{M}$ converges to matrix $\boldsymbol{C}$, since $d_i = \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[d_i]$. Since Wigner's semicircle law discusses the limit theorem where $n \to \infty$, the proof is valid if $\boldsymbol{C}$ is used instead of $\boldsymbol{M}$.

Element $C(i, j)$ of matrix $\boldsymbol{C}$ is a random variable, and is given by

$$C(i, j) = \begin{cases} \dfrac{w_{ij}}{\mathbb{E}_{p_W}[W]\sqrt{\sigma_i \sigma_j}} - \rho\sqrt{\sigma_i \sigma_j} & \text{if } L_{ij} = 1 \\ -\rho\sqrt{\sigma_i \sigma_j} & \text{otherwise} \end{cases}. \tag{26}$$

Let $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C^m(i, j)]$ be the m-th moment of $C(i, j)$. Specifically, the 1st moment $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(i, j)]$ is given by

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(i, j)] \\ &= p_{ij}\mathbb{E}_{f_{W|L_{ij}}}\left[\frac{W}{\mathbb{E}_{p_W}[W]\sqrt{\sigma_i \sigma_j}} - \rho\sqrt{\sigma_i \sigma_j}\,\middle|\, L_{ij} = 1\right] \\ &\qquad + (1 - p_{ij})\mathbb{E}_{f_{W|L_{ij}}}\left[\rho\sqrt{\sigma_i \sigma_j}\,\middle|\, L_{ij} = 0\right] \\ &= 0. \end{aligned} \tag{27}$$

For $m \geq 2$, the m-th moment $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(i, j)^m]$ is bounded by

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C^m(i, j)] \\ &= p_{ij}\mathbb{E}_{f_{W|L_{ij}}}\left[\left(\frac{W}{\mathbb{E}_{p_W}[W]\sqrt{\sigma_i \sigma_j}} - \rho\sqrt{\sigma_i \sigma_j}\right)^m \middle|\, L_{ij} = 1\right] \\ &\qquad + (1 - p_{ij})\mathbb{E}_{f_{W|L_{ij}}}\left[\left(\rho\sqrt{\sigma_i \sigma_j}\right)^m \middle|\, L_{ij} = 0\right] \\ &= p_{ij}\sum_{l=0}^{m-2}\binom{m}{l}\frac{(-p_{ij})^l\mathbb{E}_{p_W}[W^{m-l}]}{\mathbb{E}_{p_W}[W]^{m-l}(\sigma_i \sigma_j)^{m/2}} \\ &= \rho\sum_{l=0}^{m-2}\binom{m}{l}\frac{(-\rho)^l\mathbb{E}_{p_W}[W^{m-l}]}{\mathbb{E}_{p_W}[W]^{m-l}(\sigma_i \sigma_j)^{m/2-l-1}} \qquad (28) \\ &= \frac{\rho\,\mathbb{E}_{p_W}[W^m]}{\mathbb{E}_{p_W}[W]^m(\sigma_i \sigma_j)^{m/2-1}}\,(1 + o(1)) \\ &\leq \frac{\rho\,\mathbb{E}_{p_W}[W^m]}{\mathbb{E}_{p_W}[W]^m\sigma_{\min}^{m-2}}\,(1 + o(1))\,. \end{aligned} \tag{29}$$

To derive the above equation, the term with $l = 0$ in the sum since $\rho = 1/(n\,\sigma_{\text{avg}}) = 1/\sigma(n) = o(1)$ was retained. Note that $\sigma(f(n))$ is a function that increases faster than or equal to $f(n)$ as $n \to \infty$. Then, we use $f(n) = o(1)$ as

$$\lim_{n\to\infty} f(n) = 0. \tag{30}$$

The m-th moment $\mathrm{Tr}[\boldsymbol{C}^m]$ is given by

$$\begin{aligned} \mathrm{Tr}[\boldsymbol{C}^m] &= \sum_{\boldsymbol{v}_m\in\Phi_{n,m}} C(v_1, v_2)C(v_2, v_3)\ldots C(v_m, v_1) \\ &= \sum_{\boldsymbol{v}_m\in\Phi_{n,m}}\prod_{l=1}^{h(\boldsymbol{v}_m)} C(e_l)^{m_l}, \end{aligned} \tag{31}$$

where $\boldsymbol{v}_m = (v_1, v_2, v_3, \ldots, v_m)$ represents a cycle of length m in the complete graph with n nodes, and $\Phi_{n,m}$ is the set of the cycles with length m. In cycle $\boldsymbol{v}_m$, $h(\boldsymbol{v}_m)$ is the number of disjoint links, e_l is the l-th link, and m_l is the occurrence number of link e_l. In particular, m_l satisfies

$$\sum_{l=1}^{h(\boldsymbol{v}_m)} m_l = m. \tag{32}$$

where (i, j) and (j, i) are treated as the same link in a cycle when counting m_l since $\boldsymbol{C}$ is a symmetric matrix.

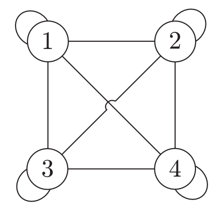


Fig. 1 The complete graph with 4 nodes.

Figure 1 shows the example of the complete graph with 4 nodes. For example, $\Phi_{4,6}$ includes $\boldsymbol{v}_6 = (1, 2, 3, 1, 2, 3)$. In cycle $\boldsymbol{v}_6$, $h(\boldsymbol{v}_6) = 3$, $e_2 = (2, 3)$ and $m_2 = 2$.

From Eq. (23) and $\boldsymbol{M} \approx \boldsymbol{C}$, the m-th moment $\mathbb{E}_{f_\Lambda^{(n)}}[(1 - \Lambda)^m]$ is approximated by

$$\mathbb{E}_{f_\Lambda^{(n)}}[(1 - \Lambda)^m] \approx \frac{1}{n-1}\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^m]]. \tag{33}$$

As $n \to \infty$, the error of the approximation approaches to 0. Hence, we investigate $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^m]]$ in order to prove that $\mathbb{E}_{f_\Lambda^{(n)}}[(1 - \Lambda)^m]$ converges to Eqs. (19) and (20) as $n \to \infty$.

To show the convergence of even moments $\mathbb{E}_{f_\Lambda^{(n)}}[(1 - \Lambda)^{2m}]$, we investigate $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}\left[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]\right]$. Using the independence of $C(i, j)$, an even moment $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}\left[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]\right]$ is bounded by

$$\begin{aligned} \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}\left[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]\right] &= \sum_{\boldsymbol{v}_m\in\Phi_{n,2m}}\prod_{l=1}^{h(\boldsymbol{v}_m)}\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(e_l)^{m_l}] \\ &\leq \sum_{l=0}^{m}|Z_{l,m}|\frac{\mathbb{E}_{p_W}[W^2]^l\,\mathbb{E}_{p_W}[W^{2(m-l)}]\rho^l}{\mathbb{E}_{p_W}[W]^{2m}\sigma_{\min}^{2m-2l}}\,(1 + o(1)) \\ &\leq \sum_{l=0}^{m}|Z_{l,m}|\frac{\mathbb{E}_{p_W}[W^2]^l\rho^l}{\mathbb{E}_{p_W}[W]^{2m}\sigma_{\min}^{2m-2l}}\,(1 + o(1)), \end{aligned} \tag{34}$$

where $Z_{l,m}$ is a set of cycles $(v_1, v_2, \ldots, v_{2m})$ with length $2m$ and disjoint $l + 1$ nodes. Note that $Z_{l,m} \subset \Phi_{n,2m}$. For example, $(1, 2, 3, 1, 2, 3)$ is included in $Z_{2,3}$. The cycles included in $Z_{l,m}$ are composed of at least l disjoint links. To derive the second right-hand side of the above equation, we first divide all cycles in $\Phi_{n,2m}$ into the sets $Z_{l,m}$ of the cycles that the number of disjoint links is l. Each set has $|Z_{l,m}|$ cycles. Using Eq. (29), for $\boldsymbol{v}_m \in Z_{l,m}$, we obtain

$$\prod_{l=1}^{h(\boldsymbol{v}_m)}\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(e_l)^{m_l}] \leq \frac{\mathbb{E}_{p_W}[W^2]^l\,\mathbb{E}_{p_W}[W^{2(m-l)}]\rho^l}{\mathbb{E}_{p_W}[W]^{2m}\sigma_{\min}^{2m-2l}}, \tag{35}$$

since that $\mathbb{E}_{pw}[W^2]$ is the largest in the m-th moments $\mathbb{E}_{pw}[W^m]$ for $m \geq 2$. In the first right-hand side of Eq. (34), the reason why the cycles longer than m are not considered, is as follows. First, $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(i,j)] = 0$. Hence, $\prod_{l=1}^{h(\boldsymbol{v}_m)} \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(e_l)^{m_l}]$ is 0 except if $m_l \geq 2$ for $1 \leq l \leq h(\boldsymbol{v}_m)$. Since $m_l \geq 2$ for $1 \leq l \leq h(\boldsymbol{v}_m)$, the number of links l in the cycle is at most m.

To extract the main term of $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^{2m}]]$, we rewrote Eq. (34) with

$$\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^{2m}]] \leq \sum_{l=1}^{m} \eta_{l,m} = \eta_{m,m} \sum_{l=1}^{m} \frac{\eta_{l,m}}{\eta_{m,m}}, \tag{36}$$

where $\eta_{l,m}$ is

$$\eta_{l,m} = |Z_{l,m}| \frac{\mathbb{E}_{pw}[W^2]^l \rho^l}{\mathbb{E}_{pw}[W]^{2m} \sigma_{\min}^{2m-2l}}. \tag{37}$$

In the above equation, $|Z_{l,m}|$ is given by

$$\begin{aligned} |Z_{l,m}| &= |Z_a(l,m)|\,|Z_b(l,m)|\,|Z_c(l,m)| \\ &= \frac{n!}{(n-l-1)!} \binom{2m}{2l} (l+1)^{4(m-l)} \frac{1}{l+1} \binom{2l}{l}, \end{aligned} \tag{38}$$

where $Z_a(l,m)$ is the number of permutations $(i_k)_{1 \leq k \leq l+1}$ by selecting $l+1$ nodes from n nodes, and is given by

$$|Z_a(l,m)| = \frac{n!}{(n-l-1)!}. \tag{39}$$

Then, $|Z_b(l,m)|$ is the number of combinations such that each i_k appears more than once in the cycle with length $2m$ and is given by

$$|Z_b(l,m)| = \binom{2m}{2l} \left((l+1)^2\right)^{2(m-l)} = \binom{2m}{2l} (l+1)^{4(m-l)}. \tag{40}$$

In the above equation, since there is no restriction except that each i_k must appear at least twice, it is multiplied by $(l+1)^{4(m-l)}$. Moreover, $Z_c(l,m)$ is the number of the second appearance positions for i_k and is given by

$$|Z_c(l,m)| = \frac{1}{l+1} \binom{2l}{l}. \tag{41}$$

Note that the right side of the above equation is the Catalan number.

By substituting Eq. (38) into Eq. (37), $\eta_{l,m}$ is given by

$$\begin{aligned} \eta_{l,m} &= |Z_{l,m}| \frac{\mathbb{E}_{pw}[W^2]^l \rho^l}{\mathbb{E}_{pw}[W]^{2m} \sigma_{\min}^{2m-2l}} \\ &= \frac{n!}{(n-l-1)!} \binom{2m}{2l} (l+1)^{4(m-l)} \\ &\qquad \frac{1}{l+1} \binom{2l}{l} \frac{\mathbb{E}_{pw}[W^2]^l \rho^l}{\mathbb{E}_{pw}[W]^{2m} \sigma_{\min}^{2m-2l}}. \end{aligned} \tag{42}$$

Then, $\frac{\eta_{l,m}}{\eta_{m,m}}$ in (36) is bounded by

$$\begin{aligned} \frac{\eta_{l,m}}{\eta_{m,m}} &= \frac{(n-m-1)!}{(n-l-1!)} \binom{2m}{2l} (l+1)^{4(m-l)} \\ &\qquad \frac{m+1}{l+1} \frac{\binom{2l}{l}}{\binom{2m}{m}} \frac{\mathbb{E}_{pw}[W^2]^l \mathbb{E}_{pw}[W^{2m-2l}] \rho^l}{\sigma_{\min}^{2m-2l} \mathbb{E}_{pw}[W^2]^m \rho^m} \\ &\leq \binom{2m}{2l} \frac{n^{l-m} (l+1)^{4(m-l)} 4^{l-m}}{\sigma_{\min}^{2(m-l)} \mathbb{E}_{pw}[W^2]^{m-l} \rho^{m-l}} \\ &\leq \frac{n^{l-m}\, 2\, m^{2(m-l)}\, m^{4(m-l)} 4^{l-m}}{\sigma_{\min}^{2(m-l)} \mathbb{E}_{pw}[W^2]^{m-l} \rho^{m-l}} \\ &\leq 2 \left[\frac{\sigma_{\mathrm{avg}}\, m^6}{4\, \sigma_{\min}^2 \mathbb{E}_{pw}[W^2]} \right]^{m-l}. \end{aligned} \tag{43}$$

To obtain the upper bound of $\frac{\eta_{l,m}}{\eta_{m,m}}$, the Stirling's approximation was used, and is given by

$$\sqrt{2\pi} n^{n+1/2} e^{-n} \leq n! \leq n^{n+1/2} e^{-n+1}. \tag{44}$$

According to the following discussion, if the degree condition is satisfied, the right-hand side of Eq. (43) for $m > l$ becomes $o(1)$. First, for $m > l$, we derive

$$\left[\frac{\sigma_{\mathrm{avg}}\, m^6}{4\, \sigma_{\min}^2 \mathbb{E}_{pw}[W^2]} \right]^{m-l} = o(1). \tag{45}$$

To prove the above equation, we should show

$$\frac{\sigma_{\mathrm{avg}}\, m^6}{4\, \sigma_{\min}^2 \mathbb{E}_{pw}[W^2]} = o(1). \tag{46}$$

Similarly for [9], using $m = \log n$, we obtain the condition

$$\sigma_{\min}^2 = \sigma((\log n)^6) \frac{\sigma_{\mathrm{avg}}}{\mathbb{E}_{pw}[W^2]}, \tag{47}$$

to satisfy Eqs. (45) and (46). The condition (47) is the same as the degree condition (16).

Therefore, if the degree condition (16) is satisfied, for $m > l$, we obtain

$$\frac{\eta_{l,m}}{\eta_{m,m}} = o(1). \tag{48}$$

By substituting the above equation into Eq. (36), the upper bound of $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^{2m}]]$ is given by

$$\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^{2m}]] \leq (1 + o(1))\, \eta_{m,m}. \tag{49}$$

On the other hand, the lower bound of $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^{2m}]]$ is given by

$$\begin{aligned} \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}[\boldsymbol{C}^{2m}]] &= \sum_{\boldsymbol{v}_m \in \Phi_{n,m}} \prod_{h=1}^{h(\boldsymbol{v}_m)} \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(e_l)^{m_l}] \\ &\geq |Z_{m,m}| \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(i,j)^2]^m \end{aligned}$$

$$\geq |Z_{m,m}|\rho^m \frac{\mathbb{E}_{pw}[W^2]^m}{\mathbb{E}_{pw}[W]^{2m}}(1-o(1))$$
$$= (1-o(1))\,\eta_{m,m}. \tag{50}$$

Note that the second right-hand side of the above equation was obtained by only using the cycles with disjoint m links. To derive the third right-hand side, we used

$$\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(i,j)^2] \geq \rho \frac{\mathbb{E}_{pw}[W^2]}{\mathbb{E}_{pw}[W]^2}(1-o(1)). \tag{51}$$

This can be derived from Eq. (28).

According to Eqs. (36) and (50), $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]]$ is bounded by

$$\eta_{m,m}(1-o(1)) \leq \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]] \leq \eta_{m,m}(1+o(1)). \tag{52}$$

As $n \to \infty$, the even moment $\mathbb{E}_{f_\Lambda^{(n)}}[(1-\lambda)^{2m}]$ converges to

$$\lim_{n\to\infty} \mathbb{E}_{f_\Lambda^{(n)}}[(1-\lambda)^{2m}] = \lim_{n\to\infty} \frac{1}{n-1}\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]]$$
$$= \lim_{n\to\infty} \frac{1 \pm o(1)}{n-1}\eta_{m,m}$$
$$= \lim_{n\to\infty} \frac{n!}{(n-1)(n-m-1)!}\frac{1}{m+1}\binom{2m}{m}\frac{\mathbb{E}_{pw}[W^2]^m \rho^m}{\mathbb{E}_{pw}[W]^{2m}}$$
$$= \lim_{n\to\infty} g(n)\frac{\mathbb{E}_{pw}[W^2]^m}{\sigma_{\mathrm{avg}}^m \mathbb{E}_{pw}[W]^{2m}}\frac{1}{m+1}\binom{2m}{m}$$
$$= \frac{\mathbb{E}_{pw}[W^2]^m}{\sigma_{\mathrm{avg}}^m \mathbb{E}_{pw}[W]^{2m}}\frac{1}{m+1}\binom{2m}{m}$$
$$= \left(\frac{\tilde{r}}{2}\right)^{2m}\frac{(2\,m)!}{m!\,(m+1)!}, \tag{53}$$

where $g(n)$ is defined by

$$g(n) := \frac{n!}{(n-1)(n-m-1)!\,n^m}. \tag{54}$$

Note that $\lim_{n\to\infty} g(n) = 1$.

Therefore, if a weighted network G satisfies the degree condition (16), the even moment $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]]$ converges to $\mathbb{E}_{\tilde{f}_\Lambda}[(1-\lambda)^{2m}]$ as $n \to \infty$.

Next, we investigate the odd moment $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m+1}\right]]$. Similar to the even moment $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m}\right]]$, $\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m+1}\right]]$ is bounded by

$$\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m+1}\right]] = \sum_{\boldsymbol{v}_m \in \Phi_{n,2m+1}} \prod_{l=1}^{h(\boldsymbol{v}_m)} \mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[C(e_l)^{m_l}]$$
$$\leq \sum_{l=0}^{m} |Z_{l,m}| \frac{\mathbb{E}_{pw}[W^2]^{l-1}\rho^l}{\mathbb{E}_{pw}[W]^{2m+1}\sigma_{\min}^{2m-2l+1}}(1+o(1))$$
$$\leq \sum_{l=0}^{m} \frac{\eta'_{l,m}}{\sigma_{\min}} = \eta'_{m,m}\sum_{l=0}^{m}\left(\frac{\eta'_{l,m}}{\sigma_{\min}\,\eta'_{m,m}}\right), \tag{55}$$

where $\sum_{h=1}^{l} m_h = 2m+1$, and $\eta'_{l,m}$ is given by

$$\eta'_{l,m} = |Z_{l,m}|\frac{\mathbb{E}_{pw}[W^2]^{l-1}\rho^l}{\mathbb{E}_{pw}[W]^{2m+1}\sigma_{\min}^{2m-2l}}(1+o(1)). \tag{56}$$

Using a method analogous to the one for the even moment, $\eta'_{l,m}/(\sigma_{\min}\,\eta'_{m,m})$ is bounded by

$$\frac{\eta'_{l,m}}{\sigma_{\min}\,\eta'_{m,m}} \leq \frac{4\sqrt{\mathbb{E}_{pw}[W^2]}}{\sqrt{\sigma_{\mathrm{avg}} m^3}}\left[\frac{\sigma_{\mathrm{avg}}\,m^6}{4\,\sigma_{\min}^2 \mathbb{E}_{pw}[W^2]}\right]^{m-l+1/2}$$
$$\leq 4\sqrt{\mathbb{E}_{pw}[W^2]}\left[\frac{\sigma_{\mathrm{avg}}\,m^6}{4\,\sigma_{\min}^2 \mathbb{E}_{pw}[W^2]}\right]^{m-l+1/2}. \tag{57}$$

If the degree condition (16) is satisfied, for $m \geq l$, we obtain

$$\left[\frac{\sigma_{\mathrm{avg}}\,m^6}{4\,\sigma_{\min}^2\,\mathbb{E}_{pw}[W^2]}\right]^{m-l+1/2} = o(1). \tag{58}$$

Hence, the upper bound of $1/(n-1)\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m+1}\right]]$ is given by

$$\frac{1}{n-1}\mathbb{E}_{\mathbb{P}[\boldsymbol{\Gamma}]}[\mathrm{Tr}\left[\boldsymbol{C}^{2m+1}\right]] \leq \frac{o(1)}{n-1}\eta'_{m,m}. \tag{59}$$

As $n \to \infty$, the right-hand side of the above equation converges to

$$\lim_{n\to\infty} \frac{o(1)}{n-1}\eta'_{m,m} = 0 \tag{60}$$

Hence, as $n \to \infty$, the odd moment $\mathbb{E}_{f_\Lambda^{(n)}}[(1-\Lambda)^{2m+1}]$ is given by

$$\lim_{n\to\infty} \mathbb{E}_{f_\Lambda^{(n)}}[(1-\Lambda)^{2m+1}] = \lim_{n\to\infty} \frac{o(1)}{n-1}\eta'_{m,m}$$
$$= 0. \tag{61}$$

Therefore, if a weighted network G satisfies the degree condition (16), the odd moment also converges to $\mathbb{E}_{\tilde{f}_\Lambda}[(1-\Lambda)^{2m+1}]$ as $n \to \infty$. □

4. Numerical Example

We confirm the validity of the degree condition (16) and Eq. (18) in Wigner's semicircle law for a weighted network G derived in Sect. 3. For brevity, we only show the results using the BA model [13], which is a well-known random network generation model.

Similar to [6], we generate a weighted network G with the following procedures, based on the BA model.

1. Generate an unweighted network with n nodes and average degree $\sigma_{\mathrm{avg}}^{\mathrm{BA}}$, according to the BA model.
2. Randomly cut the links of the unweighted network until

Table 1 Parameter configuration.

Average degree of unweighted BA network, $\sigma_{\rm avg}^{(BA)}$	40
Number of nodes of the weighted network G, n	1,000
Average degree of the weighted network G, $\sigma_{\rm avg}$	20
Pareto index of the Pareto distribution, α	3
Number of bins, n_h	50

the average degree is $\sigma_{\rm avg}$. This procedure prevents minimum degree $\sigma_{\rm min}$ from being fixed, and does not lose the scale-free property of the BA network [6].

3. Randomly assign the weight of each link using the Pareto distribution $p_W^{\rm par}(w)$ with $\mathbb{E}_{p_W}[W] = 1$. Note that $p_W^{\rm par}(w) \propto \alpha\, w^{-\alpha-1}$.
4. Divide the weight of each link by the maximum link weight $w_{\rm max}$. By performing this procedure, the link weight is lesser than or equal to 1, and its distribution satisfies the condition of $p_W(w)$ used in Sect. 3. Note that this procedure does not change the normalized Laplacian matrix $\boldsymbol{N}$.

In order to confirm the validity of the degree condition (16) in Wigner's semicircle law for a weighted network G, we compare eigenvalue density $f_\Lambda^{(n)}(\lambda)$ of the normalized Laplacian matrix $\boldsymbol{N}$ and semicircle distribution $\tilde{f}_\Lambda(\lambda)$. To evaluate their difference, we use the relative error $\epsilon_{\rm d}$, which is defined by

$$\epsilon_{\rm d} := \frac{1}{n_h}\sum_{i=1}^{n_h}\frac{|F_n(\theta_i) - F^*(\theta_i)|}{F(\theta_i)}, \tag{62}$$

where $\theta_i = (i-1/2)h_b + \lambda_2$ and $h_b = (\lambda_n - \lambda_2)/n_h$. In (62), $F_n(\theta_i)$ is the value obtained by dividing the number of eigenvalues of $\boldsymbol{N}$ in the interval $[\theta_i - h_b/2, \theta_i + h_b/2]$ by $n-1$. On the contrary, $F^*(\theta_i)$ is the value of the integral of $\tilde{f}_\Lambda(\lambda)$ within $[\theta_i - h_b/2, \theta_i + h_b/2]$.

To confirm the validity of Eq. (18), we compare the spectral radius r of the normalized Laplacian matrix $\boldsymbol{N}$ and $\tilde{r}$ calculated by Eq. (18). For these comparisons, we use relative error ϵ_r, which is defined by

$$\epsilon_r := \frac{|\tilde{r} - r|}{r}. \tag{63}$$

In the numerical example, the parameter configuration shown in Table 1 is used as a default parameter configuration.

First, we confirm the relationship between the characteristics of the weighted network G generated in the above procedure and the degree condition (16). Figure 2 shows the square of the minimum degree, $\sigma_{\rm min}^2$ for varying values of average degree $\sigma_{\rm avg}$. Additionally, in this figure, we plot the results with the average order $\sigma_{\rm avg}$ on the y axis for comparison. From these results, as $\sigma_{\rm avg}$ increases, the difference between $\sigma_{\rm min}^2$ and $\sigma_{\rm avg}$ increases, and hence the degree condition (16) is readily satisfied. Figure 3 shows the second moment $\mathbb{E}_{p_W}[W^2]$ of link weights for varying values of the Pareto index α. From this figure, as α increases, $\mathbb{E}_{p_W}[W^2]$ increases, and hence the degree condition (16) is also readily satisfied.

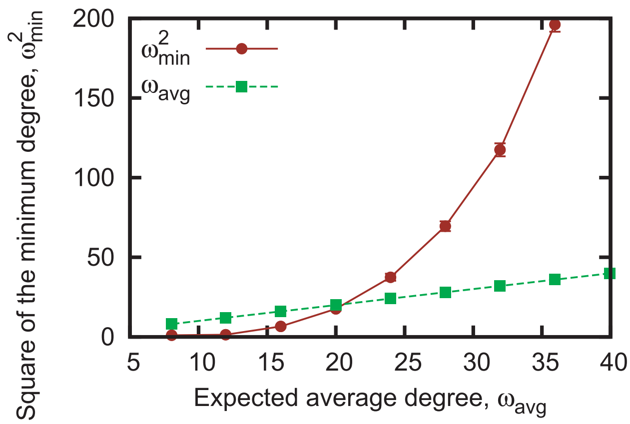


Fig. 2 Average degree $\sigma_{\rm avg}$ vs. the square of the minimum degree, $\sigma_{\rm min}^2$.

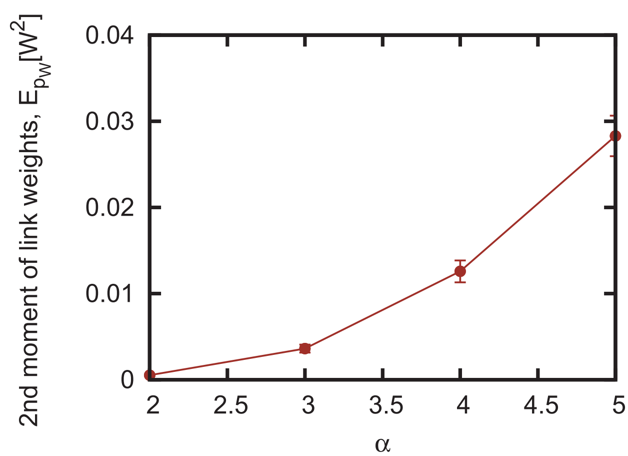
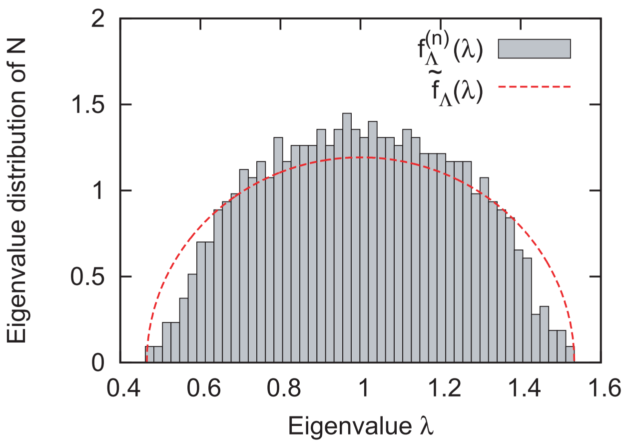


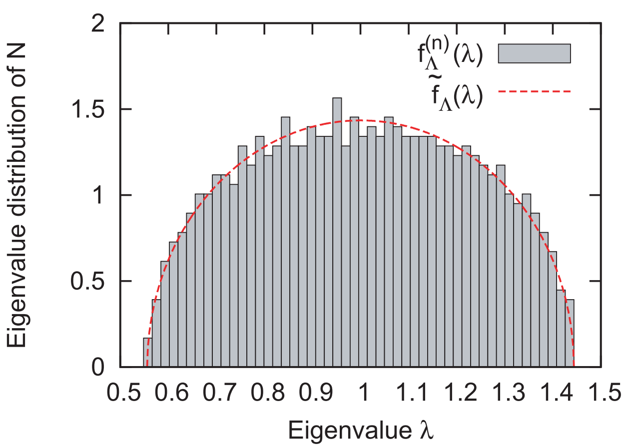
Fig. 3 Second moment $\mathbb{E}_{p_W}[W^2]$ for different pareto indices α.

Figures 4(a) and (b) show eigenvalue density $f_\Lambda^{(n)}(\lambda)$ of the normalized Laplacian matrix $\boldsymbol{N}$ (i.e, $F_n(\theta)$) and semicircle distribution $\tilde{f}_\Lambda(\lambda)$. For reference, we show the results of all link weights $w(i,j) = 1$ in Fig. 4(c). According to the results, eigenvalue density $f_\Lambda^{(n)}(\lambda)$ for $\alpha = 5$ is closer to semicircle distribution $\tilde{f}_\Lambda(\lambda)$ than that for $\alpha = 3$. This is consistent with the result of $\mathbb{E}_{p_W}[W^2]$ shown in Fig. 3. Hence, we visually confirm the validity of the degree condition (16).

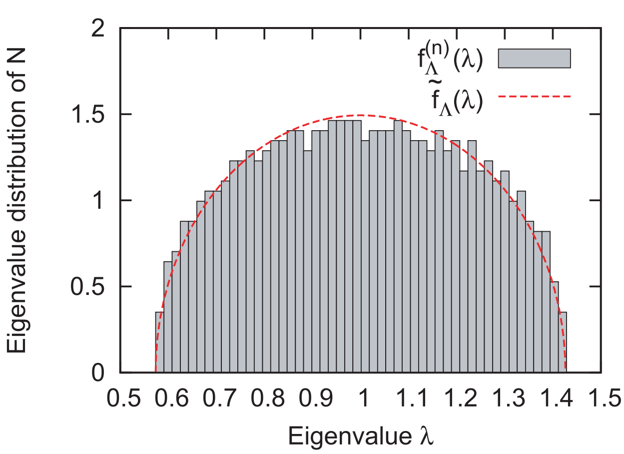
Figure 5 shows the relative error $\epsilon_{\rm d}$ of eigenvalue density $f_\Lambda^{(n)}(\lambda)$ for different average degrees $\sigma_{\rm avg}$. In this figure, we also show the result for all link weights $w(i,j) = 1$ for reference. Since n is finite, relative error $\epsilon_{\rm d}$ is not 0. We assume that if the relative error $\epsilon_{\rm d}$ is almost equal to the result for $w(i,j) = 1$, $f_\Lambda^{(n)}(\lambda)$ converges to $\tilde{f}_\Lambda(\lambda)$ as $n \to \infty$. From the results in Fig. 5, the relative error $\epsilon_{\rm d}$ decreases as average degree $\sigma_{\rm avg}$ increases, or as the Pareto index α increases. The result is consistent with the results shown in Figs. 2 and 3. Hence, the degree condition (16) is valid. Moreover, the relative error $\epsilon_{\rm d}$ for $\alpha = 5$ is almost the same as the that for $w(i,j) = 1$. Hence, if $\alpha \geq 5$, $f_\Lambda^{(n)}(\lambda)$ follows Wigner's semicircle law.



(a) pareto distribution with $\alpha = 3$



(b) pareto distribution with $\alpha = 5$



(c) $w(i, j) = 1$

Fig. 4 Eigenvalue density $f_\Lambda^{(n)}(\lambda)$ of the normalized Laplacian matrix $\boldsymbol{N}$ (i.e, $F_n(\theta)$) and semicircle distribution $\tilde{f}_\Lambda(\lambda)$.

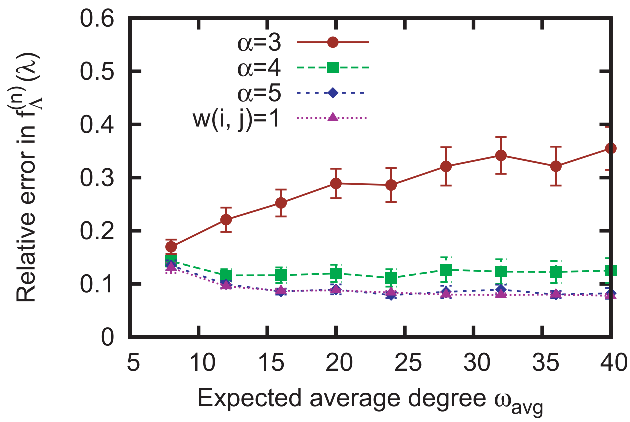


Fig. 5 Relative error ϵ_d of eigenvalue density $f_\Lambda^{(n)}(\lambda)$ for different values of average degree $\sigma_{\rm avg}$.

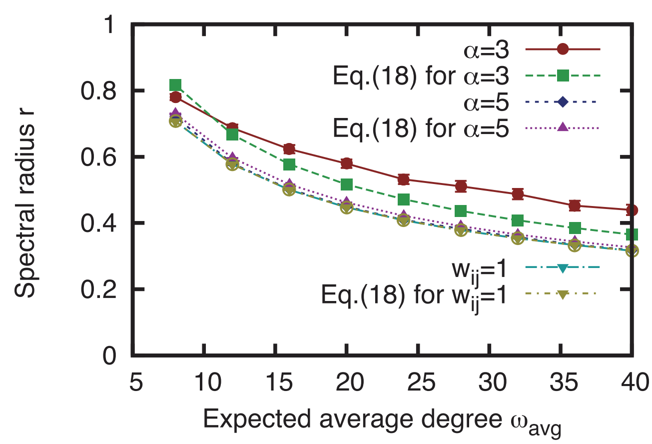


Fig. 6 Spectral radius r and $\tilde{r}$ calculated from Eq. (18) for different average degrees $\sigma_{\rm avg}$.

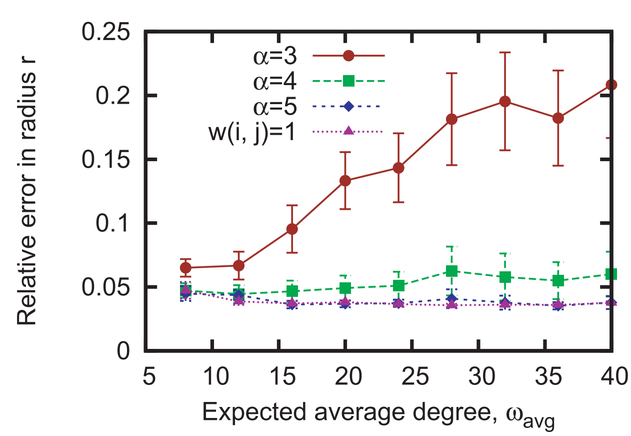


Fig. 7 Relative error ϵ_r of $\tilde{r}$ calculated from Eq. (18) for different average degrees $\sigma_{\rm avg}$.

Figure 6 shows spectral radius r and $\tilde{r}$ calculated from Eq. (18) for different average degree $\sigma_{\rm avg}$. In this figure, we also plot the result for all link weights $w(i, j) = 1$ for reference. From Fig. 6, the spectral radius r almost coincides with $\tilde{r}$ except for $\alpha = 3$. Hence, spectral radius r can be calculated accurately using Eq. (18) if the degree condition (16) is satisfied.

Figure 7 shows the relative error ϵ_r of $\tilde{r}$ calculated from Eq. (18) for different average degrees $\sigma_{\rm avg}$. In this figure, the result of all link weights $w(i, j) = 1$ is also plotted for reference. According to Fig. 7, it is clear that relative error

ϵ_r is small when the degree condition (16) is met, as with the result shown in Fig. 5. Hence, Eq. (18) is valid as the approximate expression of the spectral radius r.

From the above results, we conclude that the degree condition (16) and Eq. (18) derived in Sect. 3 are valid.

5. Related Work and Discussion

In [14], [15], weighted spectral distribution (WSD) was used to analyze a network G with the eigenvalue distribution of the normalized Laplacian matrix $\boldsymbol{N}$. In [16], a method was proposed to accelerate the calculation of WSD. WSD is given by a set of n_d elements calculated from the eigenvalues of $\boldsymbol{N}$, and it is specifically defined by

$$\left\{(1-\theta_i)^m F_G(\lambda=\theta_i) \;\middle|\; \theta_i \in \Omega_i,\ 1 \le i \le K\right\}, \tag{64}$$

where $F_G(\lambda)$ is the number of the eigenvalues in the interval including λ among n_d intervals equally spaced in $(0, 2]$. In the above equation, m is a positive integer, and Ω_i is the i-th interval $((2(i-1)/n_d, 2i/n_d]$ for $1 \le i \le n_d$. In [14], [15], on the basis of WSD, distance $\delta_m(G_1, G_2)$ between two networks G_1 and G_2 is given by

$$\delta_m(G_1, G_2) = \sum_{i=1}^{K}(1-\theta_i)^m (F_{G_1}(\lambda=\theta_i) - F_{G_2}(\lambda=\theta_i))^2. \tag{65}$$

In [15], the characteristics of the Internet topology were clarified using the distance $\delta_m(G_1, G_2)$. In this paper, we expounded upon the universality, that the eigenvalue distribution of N follows the semicircle distribution if the weighted network G satisfies the degree condition (16). The universality supplied in this paper can be applied to the analysis of WSD. For example, the semicircle distribution is used as the standard eigenvalue distribution, and we define the measure using the distance between the eigenvalue distribution of the weighted network G and the semicircle distribution as in Eq. (65). Such a measure would be useful to analyze the specialty of the weighted network G.

A power law is a property that appears in various actual networks. In many studies (e.g., [17], [18]), the relationship between the power law and network characteristics has been analyzed. In [17], the first passage time of a random walk was investigated in weighted scale-free networks, where node degrees, link weights and weight degree follow a power-law distribution. The analysis result showed that the effect of the power law on the first passage time of a random walk. In [18], the first meeting time of two random walks was analyzed. It was shown that the first meeting time is small when node degrees follow the power-law distribution. In Sect. 4, a numerical example was shown using a weighted BA network where link weights follow the power-law distribution (Pareto distribution). According to the results shown in Fig. 6, spectral radius r decreases as index α of the power-law distribution increases. In [6], the information dissemination time in social networks was shown to increase as the spectral radius r increases. From the results shown in this paper and [6], the relationship between the power law in the link weight distribution and the information dissemination time can be understood.

6. Conclusion and Future Work

In this paper, we clarified Wigner's semicircle law for a weighted network G on the basis of random matrix theory. This law indicates that if G with n nodes satisfies the degree condition (16), the eigenvalue density of the normalized Laplacian matrix $\boldsymbol{N}$ converges to the semicircle distribution determined by the approximated spectral radius $\tilde{r}$, as $n \to \infty$. Using Eq. (18), we can calculate $\tilde{r}$ from a few network statistics (the average degree, average link weight, and square average link weight). Hence, the eigenvalue distribution of $\boldsymbol{N}$ can be obtained from these network statistics without giving all the matrix elements $N(i, j)$ accurately. Our results provide a new method to analyze a weighted network G using spectral graph theory and random graph theory.

Analyzing and designing actual networks using Wigner's semicircle law, as clarified in this paper, is an interesting future study. In particular, the characteristics of the information dissemination on social networks will be investigated, and they can aid the design of the social media to control the speed of the information dissemination.

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