PAPER

Revealing of the Underlying Mechanism of Different Node Centralities Based on Oscillation Dynamics on Networks

Chisa TAKANO[†a)] ***and*** **Masaki AIDA**[††b)], ***Members***

SUMMARY In recent years, with the rapid development of the Internet and cloud computing, an enormous amount of information is exchanged on various social networking services. In order to handle and maintain such a mountain of information properly by limited resources in the network, it is very important to comprehend the dynamics for propagation of information or activity on the social network. One of many indices used by social network analysis which investigates the network structure is "node centrality". A common characteristic of conventional node centralities is that it depends on the topological structure of network and the value of node centrality does not change unless the topology changes. The network dynamics is generated by interaction between users whose strength is asymmetric in general. Network structure reflecting the asymmetric interaction between users is modeled by a directed graph, and it is described by an asymmetric matrix in matrix-based network model. In this paper, we showed an oscillation model for describing dynamics on networks generated from a certain kind of asymmetric interaction between nodes by using a symmetric matrix. Moreover, we propose a new extended index of well-known two node centralities based on the oscillation model. In addition, we show that the proposed index can describe various aspect of node centrality that considers not only the topological structure of the network, but also asymmetry of links, the distribution of source node of activity, and temporal evolution of activity propagation by properly assigning the weight of each link. The proposed model is regarded as the fundamental framework for different node centralities.
key words: *oscillation dynamics, node centrality, social network analysis*

1. Introduction

Recently, the survey "Digital in 2016" [1] for digital, social and mobile usage was reported by the social media consulting company "We Are Social". According to the survey, the number of social media users in the world is 2.31 billion, and it is expected that its number will further increase in the future. While an enormous amount of information is exchanged by many billion users through major social networking services (SNS), including Facebook, Twitter, Google+ and Instagram, information exchange manner between users heavily depends on the structure of actual social network of users. In such a situation, social network analysis which is the process of investigating the structure of social networks has been attracted attention, and dynamics for propagation of information or activity on the social network is an interesting research object [2]–[4]. For social network analysis based on graph theory, various indices have been applied in order to describe the characteristics of networks: graph diameter/radius, average path length and node density [5], degree distribution [6], [7], cluster coefficient [8]–[10], and various node centralities [11]–[14]. Most of these indices are decided by the topological structure of the network, but some of them are strongly related to user dynamics reflecting the utilization state of the network. In this paper, we focus on node centralities which are such indices.

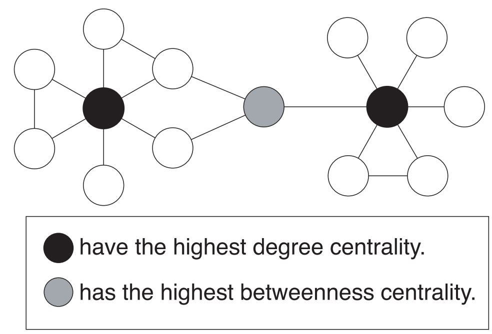

Fig. 1 Examples of node centralities.

Node centralities are indices that express the strength of importance of node. Figure 1 shows examples of typical node centrality: the degree centrality and the betweenness centrality. The black nodes have the highest degree centrality, that is, their node degree is high and they strongly contribute propagation on the network. The gray node has the highest betweenness centrality, that is, a lot of shortest paths between nodes are passing through the node and it is the essential node to relay. In this way, by choosing different measures, we can define different node centrality. Note that the node centrality is determined by the topological structure of the network. For example, it is inappropriate to judge that betweenness centrality measure of the gray node and degree centrality measure of the black node in the right subnetwork are high in the situation where the communication is performed only in the left subnetwork in the figure, because the node centrality does not reflect the utilization state of the network. In other words, the conventional node centrality is an index assuming the uniform communication on a given network structure. In general, the information exchange in social networks and the utilization state in information networks are not uniformed spatially, and origin nodes of information propagation providing new topics are

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[†]The author is with the Graduate School of Information Sciences, Hiroshima City University, Hiroshima-shi, 731-3194 Japan.
[††]The author is with the Graduate School of System Design, Tokyo Metropolitan University, Hino-shi, 191-0065 Japan.
a) E-mail: takano@hiroshima-cu.ac.jp
b) E-mail: aida@tmu.ac.jp
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unevenly distributed. Also, it is unlikely that a situation where all nodes communicate directly. In view of these facts, it is significant for practical purposes to extend the node centrality reflecting the utilization state of the network.

When a network structure is expressed algebraically by using a matrix, node-specific property and link-specific property are represented by the diagonal components and the non-diagonal components of the matrix, respectively. For this reason, an undirected/directed network is represented by a symmetric/asymmetric matrix associated with the network. In spectral graph theory, the characteristics of network structure is analyzed by studying eigenvalues and eigenvectors of the Laplacian matrix which represents the structure of the associated network [15]–[17]. The fact that the Laplacian matrix is symmetric is crucial in process to investigate its eigenvalues and eigenvectors, because symmetric matrices always can be diagonalized [18], [19]. However, various dynamics on social networks is generated as a result of the interaction between nodes (humans, computers and contents, etc.), and the strength of such interaction depends on the direction of links in general, that is, it has asymmetric link structure.

For problems mentioned above, in [20], [21], we focused on some types of link asymmetry that is reducible to represent as the characteristic of the node, and represents the structure of a directed graph with a symmetric matrix. We call the network having this type of link asymmetry as the symmetrizable directed network. Moreover, we considered the model of oscillation dynamics on symmetrizable directed networks and analyzed oscillation dynamics on symmetrizable directed networks which describes the propagation of some activities on the networks through the networks. In [22], we proposed a new index that is an extension of the conventional node centralities (degree and betweenness centralities). Unlike conventional node centralities, our proposed index can reflect various network situations including topological structure of network, asymmetry of links, the distribution of source node of activity, and temporal evolution of activity propagation. These characteristics have practical advantages. Note that the simulation evaluation in [22] focuses on the simple network model which has undirected and non-weighted links. In this paper, we discuss generally extension of node centralities based on the oscillation model on symmetrizable directed networks. In addition, we show that our proposed index can describe various node centralities by properly assigning the weight of each link, assuming the directed and weighted network model through simulation experiments. Moreover, we show that the extension of our proposed index can express the temporal evolution of activity propagation and the damped case with time.

The rest of this paper is organized as follows: First, in Sect. 2, we show the outlines of some conventional node centralities and take a brief look at applications of them for the related work. For preliminary of our study, in Sect. 3, after definition of the Laplacian matrix for symmetrizable directed networks, we introduce the scaled Laplacian matrix for describing asymmetric node interactions by a symmetric matrix [20], [21]. In Sect. 4, we analyze the solutions of the equation of motion for non-damped and damped oscillation model that can describe dynamics on networks. In addition, we mathematically clarify the relation between the proposed indices by oscillation model and several node centralities in Sect. 5. As demonstration, we compare the proposed indices by the oscillation model with several conventional node centralities by the simulation and show the validity of the proposed model. In Sect. 6, we extend the proposed index so as to describe the temporal evolution of activity propagation and express the damped dynamics on networks. Finally, we conclude this paper in Sect. 7.

2. Related Works for Node Centralities

In this section, we outline some conventional node centralities and show application examples to control and management of information networks.

Node centralities are indices of which nodes are more central than the others. The idea of first classic centralities as applied to social network and human communication was introduced by Bavelas [23], [24]. Moreover, Freeman [11] categorized three measures of node centralities based on these three features (it has more links, it can reach all the others more quickly, and it controls the flow between the others) as follows:

- Degree centrality : It is the simplest measure of the node centralities and is defined as the number of links that a node has [23]. Degree centrality has generally been extended to the sum of link weights for the weighted networks which has the weighted directed links [26]–[29].
- Closeness centrality : Closeness was defined in connected graph by [23]. It is a centrality measure which is calculated as the inverse of the sum (or average) of the length of the shortest paths between the node and all other nodes [30]. For disconnected network, the extension of closeness has been proposed which uses the harmonic mean of distances rather than the arithmetic mean [29], [31], [32]. Moreover, the directed and weighted network was studied by [27], [33].
- Betweenness centrality : Betweenness centrality is defined based on the number of shortest paths from all vertices to all others that pass through that node [23]. The importance of this conception of node centrality is in the potential of a node for control of information flow in the network. It is similar to the stress centrality [34] defined as the number of shortest paths but provides a more informative centrality index. Generalization of betweenness centrality to directed graphs and weighted graphs is shown in [29], [35]. Moreover, [36] proposed a new algorithm for calculating betweenness faster.

One of node centralities other than the above Freeman's categorized measures is the eigenvector centrality [37] based on a eigenvector for the greatest eigenvalue of the adjacency matrix. Bonacich power centrality [38] and PageRank [39] can be viewed as modifications of the eigenvector centrality. For applications of node centralities, the concepts of many proposed node centralities are used by identifying the most influential (key) persons in a social network [40], [41] and super-spreaders of disease [42], [43] as well as proposal of control algorithm of infrastructures in information network (e.g. ad hoc network and sensor network) [44]–[47].

3. Symmetric Scaled Laplacian Matrix for Describing the Link Asymmetry

3.1 Definition of the Laplacian Matrix

In graph theory, network structure is frequently expressed by a matrix. Let us consider a loop-free directed graph $\mathcal{G} = \mathcal{G}(V, E)$ with n nodes: $V = \{1, 2, \ldots, n\}$ is the set of nodes and E is the set of directed links. The directed link from node i to node j is expressed by $(i \to j) \in E$. In addition, let the link weight for link $(i \to j)$ be $w_{ij} > 0$. Then, we define the $n \times n$ square matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ as follows:

$$\mathcal{A}_{ij} := \begin{cases} w_{ij} & ((i \to j) \in E), \\ 0 & ((i \to j) \notin E). \end{cases} \tag{1}$$

This matrix represents link presence and its weights, and is called the (weighted) adjacency matrix. If $w_{ij} = w_{ji}$ for all i and j, $\mathcal{G}$ is a undirected graph, and $\mathcal{A}$ is a symmetric matrix. The adjacency matrix can be used to investigate the network structure algebraically. If $w_{ij} = 1$ for all i and j, the (i, j) component of $\mathcal{A}^k$ shows the number of paths with the length of k from node i to node j.

Next, we define the weighted out-degree d_i of node i $(i = 1, \ldots, n)$ as

$$d_i := \sum_{j \in \partial i} w_{ij}, \tag{2}$$

where ∂i denotes the set of nodes adjacent to node i. Degree matrix $\mathcal{D}$ of the weighted out-degree is defined as

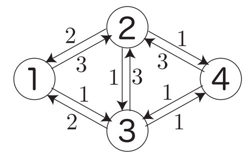
$$\mathcal{D} := \mathrm{diag}(d_1, d_2, \ldots, d_n).$$

If all link weights are $w_{ij} = 1$, d_i denotes out-degree, i.e. the number of outgoing links from node i.

Based on the above preparation, we define the Laplacian matrix $\mathcal{L}$ of the directed graph $\mathcal{G}$ as follows:

$$\mathcal{L} := \mathcal{D} - \mathcal{A}. \tag{3}$$

An example of the Laplacian matrix for an asymmetric graph is shown in Fig. 2.



$$\mathcal{A} = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}$$

$$\mathcal{D} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \mathcal{L} = \begin{bmatrix} 4 & -3 & -1 & 0 \\ -2 & 4 & -1 & -1 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$

Fig. 2 Example of Laplacian matrix.

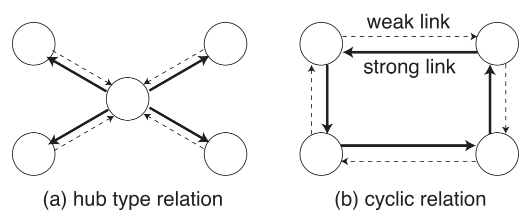


Fig. 3 Typical examples of asymmetric interaction between nodes.

3.2 Symmetrization of Laplacian Matrix and the Scaled Laplacian Matrix

Although the Laplacian matrix $\mathcal{L}$ for directed graph is generally the asymmetric matrix, we can classify the link asymmetry into two types. Figure 3 shows typical examples of link asymmetry. (a) shows a hub type relation, for example the relation of a major blogger and its followers. The link asymmetry in (a) can be expressed by the node characteristic which is the strength of node. The node characteristic means a numerical representation of the strength of node and is a consistent value. On the other hand, (b) shows a cyclic relation like rock-paper-scissors. In this case, it is not possible to define the consistent value of the strength of nodes (it is unknown which node is the strongest/weakest). Therefore, the relation of (b) is defined by not the node characteristic but the pure link characteristic. Let us consider the conditions that the asymmetric relation is classified in (a) which can be expressed by using node characteristics.

The Laplacian matrix $\mathcal{L}$ has the left eigenvector ${}^t\boldsymbol{m}$ associated with the left eigenvalue 0, that is,

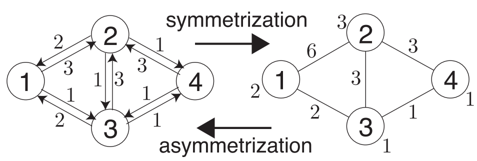
$${}^t\boldsymbol{m}\, \mathcal{L} = 0. \tag{4}$$

For each component m_i of the left eigenvector ${}^t\boldsymbol{m} = (m_1, \ldots, m_n)$, we assume the following condition:

$$m_i\, w_{ij} = m_j\, w_{ji} \quad (\equiv k_{ij}), \tag{5}$$

where $m_i > 0$. If the condition (5) is satisfied, the network is classified into the type (a) in Fig. 3 and the link asymmetry in directed network can be reduced to node characteristics in an undirected graph. We call a network satisfying (5) as a symmetrizable directed network. Here, m_i represents the node characteristic (node strength) of the node i, and w_{ij} represents the link weight from node i to node j in a directed network. The physical meaning of this condition (5) will be discussed by the oscillation model for describing dynamics on networks in the next section.

For symmetrizable directed netwrorks, the link asymmetry of $\mathcal{L}$ can be expressed by using a symmetric Laplacian matrix, and the procedure is as follows. First, we introduce



$$\mathcal{L} = \begin{bmatrix} 4 & -3 & -1 & 0 \\ -2 & 4 & -1 & -1 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -6/2 & -2/2 & 0 \\ -6/3 & 4 & -3/3 & -3/3 \\ -2/1 & -3/1 & 6 & -1/1 \\ 0 & -3/1 & -1/1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 & -2 & 0 \\ -6 & 12 & -3 & -3 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$

Fig. 4 Example of symmetrization of Laplacian matrix.

a symmetric Laplacian matrix $\boldsymbol{L}$ as $\boldsymbol{L} := \boldsymbol{D} - \boldsymbol{A}$ for an undirected network as follows: The adjacency matrix $\boldsymbol{A} = [A_{ij}]$ is defined as

$$A_{ij} := \begin{cases} k_{ij} & ((i, j) \in E), \\ 0 & ((i, j) \notin E), \end{cases} \tag{6}$$

and the degree matrix $\boldsymbol{D}$ is expressed as

$$\boldsymbol{D} = \mathrm{diag}\left(\sum_{j=1}^{n} A_{1j},\ \sum_{j=1}^{n} A_{2j},\ \ldots,\ \sum_{j=1}^{n} A_{nj} \right).$$

Since $k_{ij} = k_{ji}$, $\boldsymbol{L}$ is a symmetric Laplacian matrix for a certain undirected graph. By using $\boldsymbol{L}$, the asymmetric Laplacian matrix $\boldsymbol{\mathcal{L}}$ is expressed as

$$\boldsymbol{\mathcal{L}} = \boldsymbol{M}^{-1}\,\boldsymbol{L}, \tag{7}$$

where the scaling factors $\boldsymbol{M}$ of nodes is defined as $\boldsymbol{M} := \mathrm{diag}(m_1, m_2, \ldots, m_n)$ to reduce the link asymmetry to the characteristics of the node. Figure 4 shows a simple example of the procedure which leads to (7).

Next, we define the scaled Laplacian matrix as

$$\boldsymbol{S} := \boldsymbol{M}^{-1/2}\,\boldsymbol{L}\,\boldsymbol{M}^{-1/2}. \tag{8}$$

Note that $\boldsymbol{S}$ is a symmetric matrix as well as $\boldsymbol{L}$.

By multiplying $M^{1/2}$ to the right eigenvalue equation of $\boldsymbol{\mathcal{L}}$,

$$\boldsymbol{\mathcal{L}}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$

from the left, we have

$$\boldsymbol{M}^{1/2}\,\boldsymbol{\mathcal{L}}\,\boldsymbol{x} = \boldsymbol{S}\,(\boldsymbol{M}^{1/2}\boldsymbol{x}) = \lambda\,(\boldsymbol{M}^{1/2}\boldsymbol{x}). \tag{9}$$

This means that the scaled Laplacian matrix $\boldsymbol{S}$ has the same eigenvalues of $\boldsymbol{\mathcal{L}}$ and its eigenvector is expressed as $\boldsymbol{y} := M^{1/2}\,\boldsymbol{x}$. Since the quadratic form of $\boldsymbol{S}$ is

$${}^{t}\boldsymbol{y}\,\boldsymbol{S}\,\boldsymbol{y} = \sum_{(i,j)\in E} k_{ij} \left(\frac{y_i}{\sqrt{m_i}} - \frac{y_j}{\sqrt{m_j}} \right)^2 \geq 0,$$

the eigenvalues of $\boldsymbol{S}$ are nonnegative. We sort the eigenvalues in ascending order as $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. We can choose the eigenbasis $\boldsymbol{v}_\mu$ ($\mu = 0, 1, \ldots, n-1$) as the eigenvector of $\boldsymbol{S}$ with length of 1 associated with λ_μ, which satisfy

$$\boldsymbol{S}\,\boldsymbol{v}_\mu = \lambda_\mu\,\boldsymbol{v}_\mu, \quad \boldsymbol{v}_\mu \cdot \boldsymbol{v}_\nu = \delta_{\mu\nu}, \tag{10}$$

where $\delta_{\mu\nu}$ denotes the Kronecker delta.

4. Oscillation Model on Networks Based on Asymmetric Interaction between Nodes

In this section, we consider the solution of the equation of motion that describes oscillation phenomena on networks.

4.1 Minimal Model Describing Nodes' Interaction

Let us consider the information propagation targeted by the oscillation model. In this paper, we do not propose models that have the characteristics of specific information propagation, but pursue the so-called minimal model. The minimal model is an universal model as simplest as possible.

The minimal model is determined by considering not only a rule that expresses the user's state but also some rules that describe the interaction between users, as follows: The state of each user is represented by a one-dimensional parameter. Although multi-dimensional parameters may be required to describe the user's state, we select the simplest model that the user's state affecting the interaction between users can be represented by one-dimensional parameter. On the other hand, as rules for interaction between users, (1) if the difference in state between the user and the adjacent user is 0, no interaction occurs (2) restoring-power acts so that the difference in state between users becomes small (3) the strength of the restoring-force is represented by a monotonically increasing function of the difference against state between users. We do not assume that the user's state is the observable quantity for the model. The interaction acts according to the state of the user and the adjacent user, and no interaction occurs when all users are in the same state. The oscillation model shown in Sect. 4.2 is linearized so that the strength of restoring-force is proportional to the difference in state between users in the minimal model. Let us consider the model linearization by using the well-known Ohm's law as an example.

Figure 5 is an example of the characteristic of the current I when the voltage V is applied to the resistance. In general, no current flows when the voltage is 0, and the current monotonically increases as the voltage increases. However, there is no proportional relationship between them. This is because the heat generation of the resistance prevents the current from flowing. However, there is an approximately proportional relationship between the current and the voltage when the voltage is not large, and this relationship is known as Ohm's law. This law means a first order approximation by Taylor expansion around $V = 0$ as a function of current I for voltage V.

Fig. 5 Example of the characteristic of the current I when the voltage V is applied to the resistance.

The idea of Ohm's law is a good example to explain the linear approximation of the strength of interaction between users. Although the strength of interaction between users may be described by using a nonlinear function of the difference in the states of users, a first order approximation is possible when the difference in state between users is not large. Therefore, it was found that the linear interaction model is an extremely universal model. Of course, the model does not hold the linearity when the characteristic of nonlinearity is very strong, but nonlinear correction can be considered based on a linear model. Thus, the oscillation model expressing the interaction between users is intended not to describe a specific information propagation but to describe the common features in wide range of user interaction models, and the pursuit of the minimal model makes this possible.

Note that there is no practical meaning unless this minimal model is linked with observable information at all. In the oscillation model derived from the minimal model, the usefulness of this minimal model is indicated by the fact that the oscillation energy of nodes can reproduce the concept of the conventional degree centrality and betweenness centrality. Therefore, all the propagation of activity on a symmetrizable directed network which has been discussed using the conventional degree centrality and betweenness centrality is understood through the oscillation model.

4.2 Non-Damped Oscillation Model on Networks

Based on the idea of the minimal model in the previous section, the equation of motion for x_i which is the state of node i can be obtained by the linear restoring-force between nodes as follows:

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = -\sum_{j\in\partial i} w_{ij}(x_i - x_j), \tag{11}$$

where w_{ij} is a proportionality coefficient indicating the strength of interaction between node i and node j, and denotes the weight of a directed graph.

Here, by using the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, (11) can be expressed as

$$\frac{\mathrm{d}^2 \boldsymbol{x}(t)}{\mathrm{d}t^2} = -\boldsymbol{\mathcal{L}}\,\boldsymbol{x}(t). \tag{12}$$

The equation of motion (12) reflects the asymmetric characteristics of links described by an asymmetric Laplacian matrix $\boldsymbol{\mathcal{L}}$.

If the directed graph can be symmetrized (that is, (5) is satisfied), we have the following wave equation as the equation of motion

$$\boldsymbol{M}\,\frac{\mathrm{d}^2 \boldsymbol{x}(t)}{\mathrm{d}t^2} = -\boldsymbol{L}\,\boldsymbol{x}(t), \tag{13}$$

where $\boldsymbol{L}$ is a symmetric Laplacian matrix of the undirected graph and the scaling factor $\boldsymbol{M}$. The model represented by (13) is equivalent to the following dynamic oscillation model.

Let the weight x_i of node i be the displacement from the equilibrium point, and let the restoring-force be proportional to the difference between the displacements of node i and its adjacent node. The displacement of the node is a one-dimensional parameter representing the user's state, and the equilibrium point is the origin of the coordinates for digitizing the state of the user and can be decided arbitrarily. Since only the difference in user's state affects the interaction, the position of the origin has no essential meaning. Figure 6 shows a representative image of our oscillation model. To represent diverse oscillating behavior, we allow the spring constant of each link to be different and the mass of each node to also be different. Let a spring constant for the link between node i and node j be the link weight $k_{ij} > 0$. In addition, we assign mass $m_i > 0$ to each node i. The node characteristic m_i in (5) corresponds to the mass of node in the oscillation model. In addition the condition (5) represents Newton's third law (about the law of action and its reaction).

The displacement of the node is generally a complex-valued function as a result of solving (12) or (13) and is not an amount that can be observed directly. However, the oscillation energy obtained by squaring the displacement leads to the concept of the traditional graph theory as the node centrality. Such aspects also appear in quantum mechanics. The wave function representing the state of the system in quantum mechanics is generally a complex-valued function and is not observable, but the squared amount is associated with observable quantities.

By introducing the vector $\boldsymbol{y}(t)$, as

$$\boldsymbol{y}(t) = \boldsymbol{M}^{1/2}\,\boldsymbol{x}(t),$$

the equation of motion (12) can be expressed as

$$\frac{\mathrm{d}^2 \boldsymbol{y}(t)}{\mathrm{d}t^2} = -\boldsymbol{S}\,\boldsymbol{y}(t). \tag{14}$$

Thus, we can describe the equation of motion for the oscillation dynamics on networks using the symmetric matrix $\boldsymbol{S}$.

In order to solve the equation of motion (14), we expand $\boldsymbol{y}(t)$ as

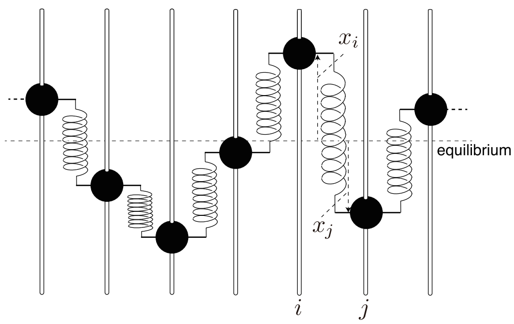


Fig. 6 Representative Image of the Oscillation Model on Symmetrizable Directed Networks.

$$\boldsymbol{y}(t) = \sum_{\mu=0}^{n-1} a_\mu(t)\, \boldsymbol{v}_\mu, \tag{15}$$

by the eigenbasis $\boldsymbol{v}_\mu$ of S. By substituting (15) into (14), we obtain n independent equations of motion for each oscillation mode μ as

$$\frac{\mathrm{d}^2 a_\mu(t)}{\mathrm{d}t^2} = -\lambda_\mu\, a_\mu(t),$$

and the solution is

$$a_\mu(t) = c_\mu\, e^{\pm \mathrm{i}\,(\omega_\mu t + \theta_\mu)}, \tag{16}$$

where c_μ is a constant, $\omega_\mu = \sqrt{\lambda_\mu}$, $\mathrm{i} = \sqrt{-1}$ and θ_μ denotes phase for the oscillation model. From the above, we have the solution of (14) as

$$\boldsymbol{y}(t) = \sum_{\mu=0}^{n-1} c_\mu\, e^{\pm \mathrm{i}\,(\omega_\mu\, t + \theta_\mu)}\, \boldsymbol{v}_\mu. \tag{17}$$

In addition, the solution of (12) is

$$\boldsymbol{x}(t) = \boldsymbol{M}^{-1/2} \left(\sum_{\mu=0}^{n-1} c_\mu\, e^{\pm \mathrm{i}\,(\omega_\mu\, t + \theta_\mu)}\, \boldsymbol{v}_\mu \right). \tag{18}$$

4.3 Damped Oscillation Model on Networks

Since an oscillation is damped with time in actual situation, we should consider the damped oscillation model. Since the damping force is usually proportional to the velocity of node and its mass, the equation of motion for the damped oscillation can be expressed as

$$\boldsymbol{M}\, \frac{\mathrm{d}^2 \boldsymbol{x}(t)}{\mathrm{d}t^2} + \boldsymbol{M}\, \gamma\, \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = -\boldsymbol{L}\, \boldsymbol{x}(t), \tag{19}$$

where $\gamma > 0$ is a constant called the damping coefficient. By using the vector $\boldsymbol{y} = \boldsymbol{M}^{1/2}\, \boldsymbol{x}$, we can symmetrize the equation of motion as

$$\frac{\mathrm{d}^2 \boldsymbol{y}(t)}{\mathrm{d}t^2} + \gamma\, \frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = -\boldsymbol{S}\, \boldsymbol{y}(t). \tag{20}$$

By applying the similar procedure using (15), we obtain n independent equations of motion for each oscillation mode μ, as

$$\frac{\mathrm{d}^2 a_\mu(t)}{\mathrm{d}t^2} + \gamma\, \frac{\mathrm{d}a_\mu(t)}{\mathrm{d}t} + \omega_\mu^2 = 0. \tag{21}$$

In order to solve (21), let us consider the characteristic equation of (21):

$$\alpha^2 + \gamma\alpha + \omega_\mu^2 = 0. \tag{22}$$

The solutions of the characteristic equation are obtained as

$$\alpha_\pm = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_\mu^2} \quad \text{(double-sign corresponds).}$$

The solutions of the equation of motion (21) are classified into three categories, depending on the solution of (22), as follows. In case that $\alpha_\pm$ are complex roots ($(\gamma/2)^2 < \omega_\mu^2$), the solution describes damped oscillations,

$$a_\mu(t) = a_\mu(0)\, e^{-(\gamma/2)t}\, e^{\pm \mathrm{i} \sqrt{\omega_\mu^2 - (\gamma/2)^2}\, t}. \tag{23}$$

In case that $\alpha_\pm$ are double roots ($(\gamma/2)^2 = \omega_\mu^2$), the solution describes the critical damping,

$$a_\mu(t) = (a_\mu(0) + c_\mu\, t)\, e^{-(\gamma/2)t}, \tag{24}$$

where c_μ is a constant. Finally, in case that $\alpha_\pm$ are two distinct real (negative) roots ($(\gamma/2)^2 > \omega_\mu^2$), the solution describes overdamping,

$$a_\mu(t) = c_\mu^+\, e^{\alpha_+ t} + c_\mu^-\, e^{\alpha_- t}, \tag{25}$$

where c_μ^+ and c_μ^- are constants.

Therefore, we obtain the solution of the equation of motion (19) for the damped oscillation as

$$\boldsymbol{x}(t) = \boldsymbol{M}^{-1/2} \left(\sum_{\mu=0}^{n-1} a_\mu(t)\, \boldsymbol{v}_\mu \right).$$

5. Oscillation Model and Node Centralities

In this section, we consider actual meaning of oscillation models on networks proposed in the previous section. First, we show the relationship between oscillation energy for each node and the node centrality, and propose the new node centrality index by using the oscillation energy. The proposed index can reproduce the conventional degree centrality and betweenness centrality as simple and special cases.

It would be inappropriate to suppose that the concept of the node displacement is unnecessary and the model can be constructed only with node centrality, even though the node centrality can be observed but the displacement of the node cannot be observed. In generalizing the node centrality as the importance of the node and considering its temporal evolution and propagation on the network, the oscillation phenomenon based on the displacement of node is necessary as the underlying structure. In addition, by considering the oscillation phenomenon, it is possible to understand the different centralities (degree centrality and betweenness centrality) in the same framework and to derive extended concept of node centrality reflecting the utilization state of the network.

5.1 Oscillation Energy and Degree Centrality

Let us consider the behavior of the solution (18) for non-damped oscillation on networks. The solution (18) is a complex valued function, and so it is difficult to interpret as an actual quantity that can be directly observed. Therefore, it is desired to make a non-negative valued index that describes the strength of activity of each node on networks.

As the first step, we introduce the oscillation energy. Since the oscillation energy of a simple harmonic oscillator for node mass m, amplitude A and angular frequency ω is $\frac{1}{2}m\,\omega^2 A^2$, the oscillation energy E_i for node i is expressed as

$$E_i = \frac{1}{2}\, m_i \sum_{\mu=0}^{n-1} \omega_\mu^2\, |a_\mu(t)|^2 \left(\frac{v_\mu(i)}{\sqrt{m_i}}\right)^2 = \frac{1}{2} \sum_{\mu=0}^{n-1} \omega_\mu^2\, |a_\mu(t)|^2\, (v_\mu(i))^2, \tag{26}$$

where $\boldsymbol{v}_\mu = {}^t(v_\mu(1), \ldots, v_\mu(n))$. Note that $v_\mu(i)$ is a real number because $\boldsymbol{S}$ is a real symmetric matrix.

As the initial condition of the wave equation (12), let us give the displacement only at a certain node. We define the node as a source node of activity. First of all, let us consider the situation that the source node of activity is chosen at random. In this case, all the oscillation modes contribute at the same strength. If we choose $|a_\mu(t)| = \sqrt{2}$ for all μ, we have

$$E_i = \sum_{\mu=0}^{n-1} \lambda_\mu\, (v_\mu(i))^2 = \frac{\sum_{j\in\partial i} k_{ij}}{m_i} = d_i. \tag{27}$$

The reason of the second equality of (27) is justified by the following discussion: $\boldsymbol{S}$ can be diagonalized by using the orthogonal matrix $\boldsymbol{P} = (\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{n-1})$ as $\boldsymbol{\Lambda} = {}^t\boldsymbol{P}\,\boldsymbol{S}\,\boldsymbol{P}$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_0, \ldots, \lambda_{n-1})$. By using $\boldsymbol{S} = \boldsymbol{P}\,\boldsymbol{\Lambda}\,{}^t\boldsymbol{P}$, we can recognized that E_i is the ith diagonal component of $\boldsymbol{S}$. Therefore, E_i gives the degree centrality of node i.

In order to demonstrate the actual meaning of the proposed index, we evaluate the oscillation energy E_i for node i using the network model shown in Fig. 7.

In this paper, we show that it is possible to derive the extended concept of node centrality from the oscillation model on the symmetrizable directed network. The property that the oscillation energy of each node results in degree centrality and betweenness centrality under specific conditions is given mathematically in the symmetrizable directed network. Therefore, the above property is established in "any" symmetrizable directed network. This paper gives some demonstrations by using a specific network model, but it is mathematically guaranteed that similar results can be obtained even for other symmetrizable directed networks. For example, the large-scale property and the scale free property which the social network has do not affect the evaluation results in this paper. Note that general directed networks that do not satisfy (5) are not considered, since this paper focuses on symmetrizable directed networks. The general directed network can also be expressed as an oscillation model, but it corresponds to the oscillation phenomenon which does not satisfy Newton's third law. In this case, the oscillation energy of the whole network can be defined, but it is generally impossible to distribute it to the oscillation energy of each node. The divergence of oscillation energy has been analyzed and discussed as a flaming phenomena in networks [48], [49].

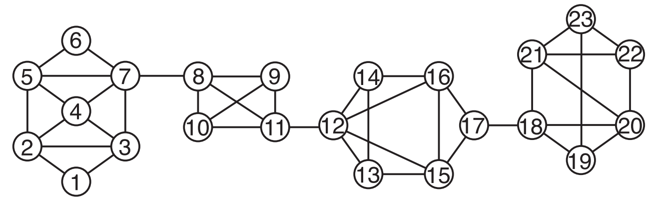


Fig. 7 Network model (undirected graph).

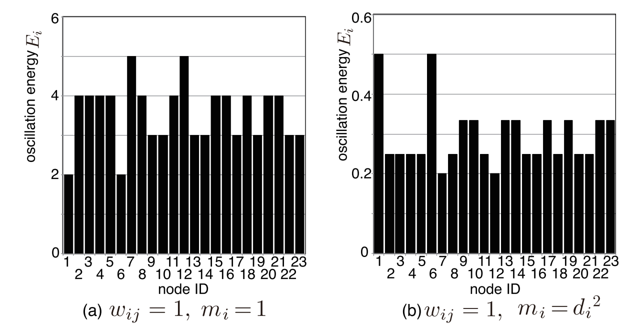


Fig. 8 Oscillation energy for each node in undirected graph ($\boldsymbol{M} = \boldsymbol{I}$, $\boldsymbol{M} = \boldsymbol{D}^2$).

Figure 8 shows the evaluation results of the oscillation energy E_i in case that the weight of all links $w_{ij} = 1$, and $|a_\mu(t)| = \sqrt{2}$ for all oscillation modes μ. The results are using mass matrix $\boldsymbol{M} = \boldsymbol{I}$ (left) and $\boldsymbol{M} = \boldsymbol{D}^2$ (right). Figure 8(a) and (b) correspond to the case that the interaction between nodes is symmetric ($\boldsymbol{M} = \boldsymbol{I}$), and an example of asymmetric node interaction ($\boldsymbol{M} = \boldsymbol{D}^2$), respectively. Therefore, Fig. 8(b) can be considered as an extension of the degree centrality for asymmetric interaction between nodes.

5.2 Oscillation Energy and Betweenness Centrality

The betweenness centrality is a well-known node centrality. While [22] has stated the betweenness centrality for an undirected graph, we explain the betweenness centrality for a directed graph to help easy understanding of characteristics of node centralities in the following subsections.

Let the number of shortest paths from node j to node k be σ_{jk}, and the number of those paths passing through the node i be $\sigma_{jk}(i)$ for the network which has the directed links.

The betweenness centrality $C_{\mathrm{bt}}(i)$ for node i is defined as

$$C_{\mathrm{bt}}(i) := \sum_{j,\,k \in V \setminus \{i\}} \frac{\sigma_{jk}(i)}{\sigma_{jk}}. \tag{28}$$

The normalized betweenness centrality $\bar{C}_{\mathrm{bt}}(i)$ is defined as

$$\bar{C}_{\mathrm{bt}}(i) := \frac{C_{\mathrm{bt}}(i)}{(n-1)\,(n-2)}. \tag{29}$$

The physical meaning of $\bar{C}_{\mathrm{bt}}(i)$ is the ratio of the number of shortest paths including node i to the number of permutations of node pairs in $V \setminus \{i\}$, that is ${}_{n-1}\mathrm{P}_2$.

Here, let us set the link weight w_{ij} as the number of the shortest paths passing through the link $(i \to j)$. Then the (weighted) degree of node i can be expressed as

$$d_i = \sum_{j \in \partial i} w_{ij} = \sum_{j,\,k \in V \setminus \{i\}} \frac{\sigma_{jk}(i)}{\sigma_{jk}} + \sum_{k \in V \setminus \{i\}} \frac{\sigma_{ik}(i)}{\sigma_{ik}}, \tag{30}$$

where $V \setminus \{i\}$ is the set of nodes excluding node i. The second equality of (30) means the sum of the number of shortest paths passing through node i and shortest paths started from node i. Since the number of shortest paths that are terminated at node i is $n-1$, we have

$$d_i = C_{\mathrm{bt}}(i) + (n-1). \tag{31}$$

Let us consider the situation that the source node of activity is chosen at random. In this case, all the oscillation modes contribute at the same strength. If we choose $|a_\mu(t)| = \sqrt{2}$ for all μ and $m_i = 1$ in (26), we have

$$E_i = d_i = C_{\mathrm{bt}}(i) + (n-1). \tag{32}$$

If there is a node that any shortest paths never pass through,

$$E_{\min} := \min_{i \in V} E_i = n - 1,$$

and that is to say that the difference $E_i - E_{\min}$ is equal to the betweenness centrality.

5.3 Typical Examples of the Correspondence between Oscillation Energy

In this subsection, we take note of the betweenness centrality for the simulation evaluation, and we clarify that the oscillation energy for the oscillation model corresponds to these centralities by properly assigning link weight between nodes as shown in Sect. 5.2.

Let us consider the directed and weighted graph for the network model used in the simulation evaluation. The network model (Fig. 9) has 23 nodes and several weighted links. The numerical character in each circle and that of beside each link denote node ID and the weight of each link, respectively. For example, the weight of the directed link from node 1 to node 2 is 1.0. We can reduce the link asymmetry in directed network to node characteristics in an undirected graph as shown in Sect. 3.2. Incidentally, Fig. 10 is the undirected network which is derived from the weighted directed network Fig. 9. We evaluate the proposed model by using the weighted directed network model as Fig. 9.

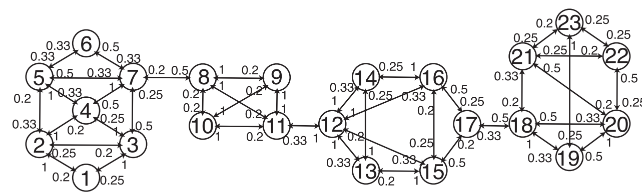


Fig. 9 Network model with weighted directed links.

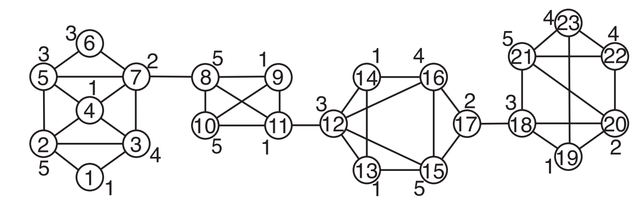


Fig. 10 Undirected network graph where the link asymmetry in weighted directed network is reduced to node characteristics.

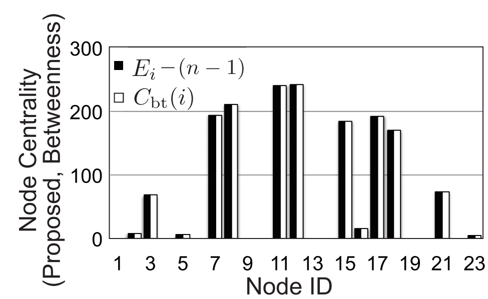


Fig. 11 Proposed index $E_i - (n-1)$ and the betweenness centrality $C_{\mathrm{bt}}(i)$ ($\gamma = 0$).

Next, we set the link weight as the number of the shortest paths passing through the link for the network model (Fig. 9) and evaluate the oscillation energy for each node. Figure 11 shows the evaluation result of the betweenness centrality C_{bt} and the energy $E_i - (n-1)$ in case that $|a_\mu(t)| = \sqrt{2}$ for all oscillation modes μ and the damping coefficient $\gamma = 0$. We can see from this figure that the energy $E_i - (n-1)$ is equal to the betweenness centrality C_{bt} for each nodes as shown in (32).

Consequently, the proposed index by the oscillation energy can describe several conventional node centralities by properly assigning each link weight, even if the topological structure of the network is asymmetric as the weighted directed network (Fig. 9).

6. Extended Node Centrality for Describing Propagation of Node Activity

6.1 Oscillation Energy for a Certain Source Node and Node Centrality

The conventional node centralities do not consider the distribution of source nodes, while they depend on the topo-

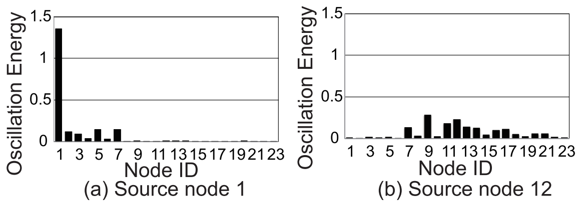


Fig. 12 Oscillation energy for each node in weighted directed graph (source node is node 1 and 12, $\gamma = 0$).

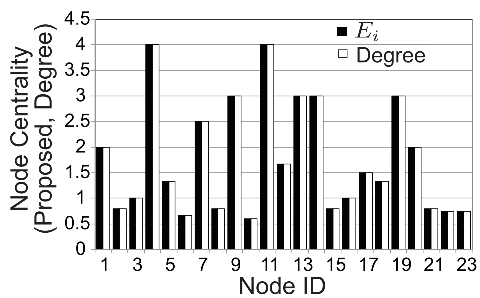


Fig. 13 Oscillation energy for each node in weighted directed graph. This shows the oscillation energy superposed for all source nodes.

logical structure of network. Generally, the real importance of nodes in a network should not be defined only by the topology, because the frequency and the amount of information distribution are different among source nodes. In this subsection, we confirm the oscillation energy for each node when a certain node is a source node of activity. We use the weighted directed network model as Fig. 9 and $\gamma = 0$. Figure 12 shows the evaluations of the oscillation energy for each node with respect to the different source node of activity. As the initial condition, a displacement of node 1 (Fig. 12(a)) or 12 (Fig. 12(b)) is set to 1 and that of the other nodes is set to 0. The results show that the oscillation energy strongly depends on a position of the source node of activity. While the number of out-going links of node 12 is larger than that of node 1, the oscillation energy in (a) is relatively larger than that in (b) because the sum of out-going link weight of node 1 (sum = 2.0) is larger than that of node 12 (sum = 1.65). Note that by superposing oscillation energy for all source nodes, we have the degree centrality of Fig. 13. Therefore, the proposed oscillation energy can be regarded as an extended index of well-known degree centrality. Consequently, the oscillation energy of each node gives extensions of the degree centrality that can reflect link asymmetry and distribution of source nodes, and corresponds to the original degree centrality in the simplest case.

6.2 Kinetic Energy and Time-Dependent Node Centrality

In the non-damped oscillation, the kinetic energy and potential energy is alternating each other with time, but the oscillation energy, which is given by the sum of the kinetic energy and potential energy, is not changed with time. So, the oscillation energy (26) does not depend on time.

However, it is necessary to consider the energy-concerning index that depends on time for the following two reasons:

- The first reason is to represent the propagation of the wave in the network. We consider the initial state where the particular node is selected as the source node. The oscillation energy of a distant node from the source node is also positive constant and does not change, even if it is before/after the arrival time of the wave. Therefore, wave propagation can not be described by using the oscillation energy.
- The second reason is the oscillating phenomenon on the network is damped with time in general. We want to describe temporal evolution of the strength of node activity.

We consider the kinetic energy of the node as a time-dependent energy-concerning index. The kinetic energy is given by $\frac{1}{2}\, m\, (\mathrm{d}x(t)/\mathrm{d}t)^2$ for the node position $x(t)$ and the mass m. If $x(t)$ is a complex valued function, there are two ways for calculation of the kinetic energy:

(i). calculation of the kinetic energy from the only real part of $\mathrm{d}x(t)/\mathrm{d}t$, that is $\frac{1}{2}\, m\, (\Re\mathfrak{e}[\mathrm{d}x(t)/\mathrm{d}t])^2$, or

(ii). calculation of the kinetic energy from a complex value $\mathrm{d}x(t)/\mathrm{d}t$, that is $\frac{1}{2}\, m\, |\mathrm{d}x(t)/\mathrm{d}t|^2$.

In the former case (i), the kinetic energy is 0 if the node stops. For the initial condition, we set that the source node stops and displacement of it is not 0 at $t = 0$. Since we want to recognize that the wave arrived at the source node at the initial time, it is necessary that the initial kinetic energy to be a positive value at the source node, even if the source node stops. In case (ii), the initial kinetic energy of the source node is positive and that is due to the contribution of the imaginary part.

From the above discussion, we propose the kinetic energy $E_i^{\mathrm{K}}(t)$ for node i at time t defined as

$$E_i^{\mathrm{K}}(t) := \frac{1}{2}\, m_i \left|\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\right|^2 = \frac{1}{2}\left|\frac{\mathrm{d}y_i(t)}{\mathrm{d}t}\right|^2,$$

as the time-dependent and energy-concerning index. Here, arbitrary constants are set by the initial conditions as follows: In the case of the damped oscillation, because all nodes stops at the time $t = 0$ and the time differential of the oscillation part of (23) is 0, $a_\mu(0)$ is a real number. Therefore, $\mathrm{d}a_\mu(t)/\mathrm{d}t|_{t=0} = 0$ for critical damping (24) and overdamping (25) brings

$$c_\mu = \frac{\gamma}{2}\, a_\mu(0),\ \ c_\mu^+ = \frac{-\alpha_-}{\alpha_+ - \alpha_-}\, a_\mu(0),\ \ c_\mu^- = \frac{\alpha_+}{\alpha_+ - \alpha_-}\, a_\mu(0).$$

Next, we evaluate the kinetic energy $E_i^{\mathrm{K}}(t)$ for each node as an extension of the degree centrality and betweenness centrality. Network model for evaluations uses the weighted directed graph shown in Fig. 9. First, we set the link weight for each link as the same value 1. Figure 14 shows the temporal evolution ($t = 1, 3, 5$) of the kinetic energy $E_i^{\mathrm{K}}(t)$

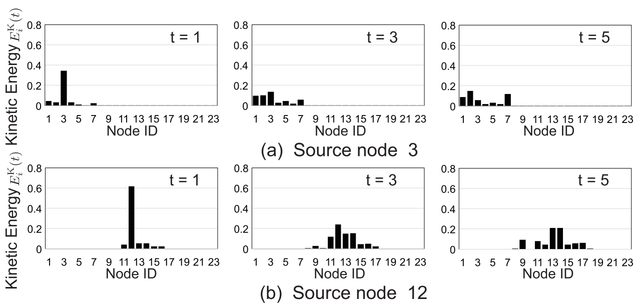


Fig. 14 Temporal evolution of the kinetic energy for the weighted directed graph (source node 3 or 12, $\gamma = 0$).

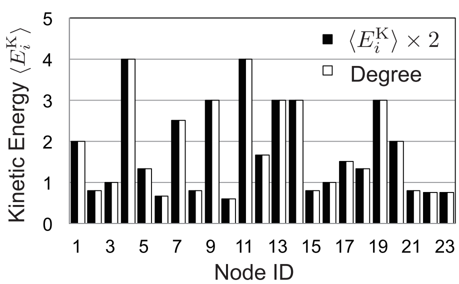


Fig. 15 $\langle E_i^{\mathrm{K}} \rangle$ superposed for all the 23 source nodes in the weighted directed graph ($\gamma = 0$).

when the displacement of the source node 3 or 12 is 1, the displacement of the other nodes is 0 for the initial condition, and the damping coefficient $\gamma = 0$. We can see from this figure that the kinetic energy $E_i^{\mathrm{K}}(t)$ expresses the behavior of wave propagation from the source node.

Next, we evaluate the average of the kinetic energy

$$\langle E_i^{\mathrm{K}} \rangle := \frac{1}{T} \int_0^T E_i^{\mathrm{K}}(t)\, \mathrm{d}t$$

for an interval T. If we set the interval $T = 1000$ and superpose the $\langle E_i^{\mathrm{K}} \rangle$ for all the 23 source nodes, the result is proportional to the degree centrality shown in Fig. 15. The value of $\langle E_i^{\mathrm{K}} \rangle$ in this figure is multiplied by a constant number (2.0) to make it easy to compare the energy $\langle E_i^{\mathrm{K}} \rangle$ with the degree centrality for each node. Let us discuss this result: In case of the non-damped oscillation model, the oscillation energy is conserved and static. As shown in Sect. 5.1, the oscillation energy E_i gives the degree centrality of node i. In addition, the oscillation energy is equal to the sum of the kinetic energy and the potential energy, and the amount of the kinetic energy is equivalent to that of the potential energy. Therefore, the degree centrality is twice as large as the kinetic energy in this case. We can see from Figs. 14 and 15 that the kinetic energy of node is an extension of the degree centrality that can describe temporal evolution of degree centrality.

Next, we set the link weight as the number of the shortest paths passing through the link for the network model (Fig. 9) and evaluate the characteristics of kinetic energy.

Figure 16 shows the temporal evolution (t = 1, 3, 5) of the kinetic energy $E_i^{\mathrm{K}}(t)$ for the source node 3 or 12

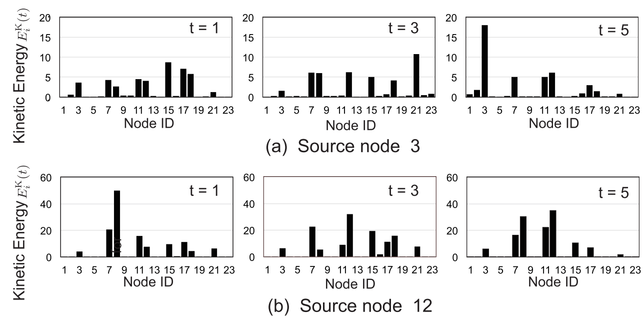


Fig. 16 Temporal evolution of the kinetic energy for the weighted directed graph (the weight for each link is the number of the shortest paths passing through the link, $\gamma = 0$).

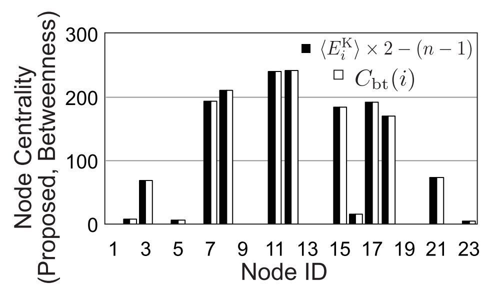


Fig. 17 $\langle E_i^{\mathrm{K}} \rangle$ superposed for all the 23 source nodes in the weighted directed graph (the weight for each link is the number of the shortest paths passing through the link, $\gamma = 0$).

for the initial condition, and Fig. 17 is the evaluation result comparing the betweenness centrality C_{bt} with the energy $\langle E_i^{\mathrm{K}} \rangle \times 2 - (n-1)$ for all the 23 source nodes. In (32), the oscillation energy E_i is equal to $C_{\mathrm{bt}}(i) + (n-1)$. The energy $\langle E_i^{\mathrm{K}} \rangle \times 2 - (n-1)$ means the oscillation energy which is double the kinetic energy minus $E_{\min}$. We can see from these figures that the energy $\langle E_i^{\mathrm{K}} \rangle \times 2 - (n-1)$ is equal to the betweenness centrality C_{bt} for each nodes and our proposed index can express the temporal evolution of C_{bt} by using the kinetic energy $\langle E_i^{\mathrm{K}} \rangle$.

6.3 Node Centrality for Different Damping Factors

Regarding the non-damped oscillation mode (the damping coefficient $\gamma = 0$), we have evaluated proposed new indices (the oscillation energy and the kinetic energy) in previous sections. In this subsection, we focus on the damped oscillation model. As shown in Sect. 4.3, the magnitude relation between the damping coefficient γ and the oscillation frequency ω_μ for each oscillation mode μ influences the solution a_μ of the equation of motion: that is, for each μ, a_μ describes (1) the damped oscillation $(\gamma/2)^2 < \omega_\mu^2$), (2) the critical damping ($(\gamma/2)^2 = \omega_\mu^2$), or (3) the overdamping ($(\gamma/2)^2 > \omega_\mu^2$). Here, for the weighted directed graph (Fig. 9), the distribution of the oscillation frequency ω_μ for each oscillation mode μ is shown in Fig. 18.

We investigate the characteristics of the oscillation energy in the damped oscillation model where has the damping coefficient $\gamma = 0.2, 0.4, 0.8$. Note that the oscillation energy

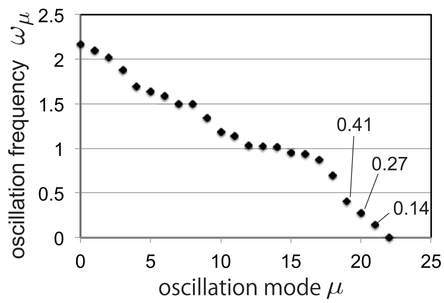


Fig. 18 Oscillation frequency ω_μ for each oscillation mode μ for the weighted directed graph.

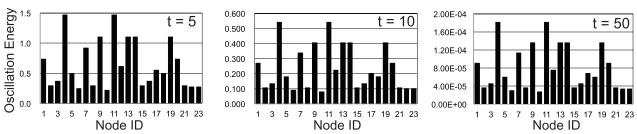


Fig. 19 Temporal evolution of oscillation energy for each node in $\gamma = 0.2$.

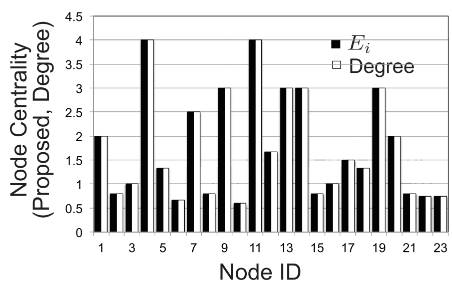


Fig. 20 Oscillation energy (multiplied by a constant number) and the degree centrality for each node for $\gamma = 0.2$, time = 50.

in previous evaluations does not change over time because the value of the damping coefficient γ is 0.

If the damping coefficient $\gamma = 0.2$, the relation $(\gamma/2)^2 < \omega_\mu^2$ $(\mu = 0, 1, 2, \cdots, 22)$ holds for almost all the oscillation mode μ (except for the minimum value 0 of ω_{22}), that is, the solution a_μ $(\mu = 0, 1, 2, \cdots, 21)$ describes damped oscillations and the solution a_{22} expresses overdamping. Figure 19 shows the temporal evolution (t = 5, 10 and 50) of the oscillation energy superposed for all source nodes, where the damping coefficient $\gamma = 0.2$ and the network model is Fig. 9. This figure shows that the behavior of the oscillation energy does not change through passage of time if the damping coefficient is small as $\gamma = 0.2$, while the value of oscillation energy for each node diminishes gradually. That is that the oscillation energy at some time is equal to the result at the other time multiplied by a constant number. If the oscillation energy is multiplied by a constant number for the result at time 50, we obtain the result that the oscillation energy corresponds to the degree centrality as shown in Fig. 20.

Next, in case of $\gamma = 0.4$ $(\gamma/2 = 0.2)$, the solution a_μ $(\mu = 0, 1, 2, \cdots, 20)$ describes damped oscillations and a_{21} and a_{22} express overdamping. Figures 21 shows the temporal evolution of oscillation energy in $\gamma = 0.4$. Since the damping coefficient γ is larger and oscillation modes of

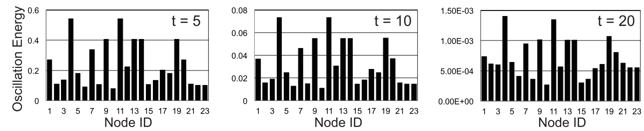


Fig. 21 Temporal evolution of oscillation energy for each node in $\gamma = 0.4$.

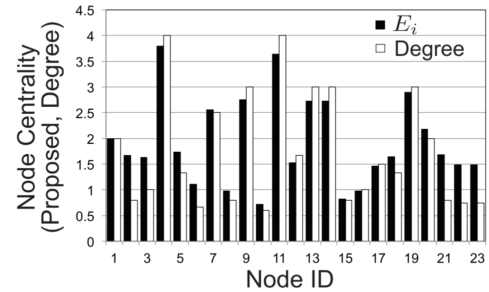


Fig. 22 Oscillation energy (multiplied by a constant number) and the degree centrality for each node for $\gamma = 0.4$, $t = 20$.

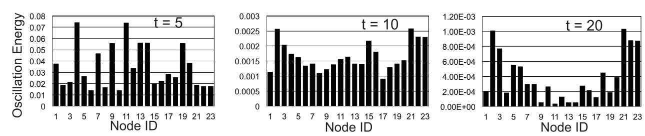


Fig. 23 Temporal evolution of oscillation energy for each node in $\gamma = 0.8$.

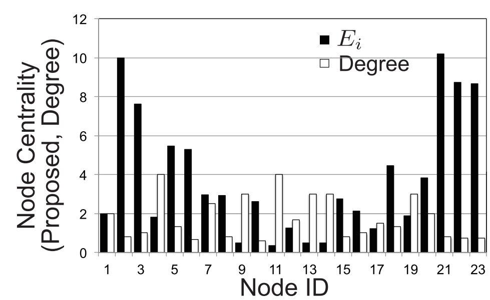


Fig. 24 Oscillation energy (multiplied by a constant number) and the degree centrality for each node for $\gamma = 0.8$, $t = 20$.

overdamping increases, the values of the oscillation energy declines more drastically than that in Fig. 19. Moreover, the relative relation of the oscillation energy for each node changes over time: the oscillation energy at some time is "not" equal to the result at the other time multiplied by a constant number. Figure 22 shows the comparison result with the degree centrality at time $t = 20$. We can see from this figure that the oscillation energy does not coincide with the degree centrality in $\gamma = 0.4$.

In addition, in case of $\gamma = 0.8$, the solution a_μ $(\mu = 0, 1, 2, \cdots, 19)$ describes damped oscillations and the other a_{20}, a_{21} and a_{22} are overdamping. Figures 23 and 24 show results in $\gamma = 0.8$ and the oscillation energy completely differs from the degree centrality. From the above, we can see that the conformity between the degree centrality and the oscillation energy depends on the value of γ. Note that we obtain the same results for comparison between the oscillation energy and the betweenness centrality, and so we omit the explanation.

7. Conclusion

This paper discusses generally extension of node centralities based on the oscillation model. For the oscillation model to describe the network dynamics, we proposed the oscillation energy and the kinetic energy of each node as new indices of node centrality. Our proposed indices by the oscillation energy and the kinetic energy can correspond to various node centralities by properly assigning the weight of each link. In addition, while the conventional node centralities (e.g. degree and betweenness) depend on the topological structure of network and the value of node centrality does not change unless the topology changes, our proposed extended indices can describe various network situations including topological structure of network, asymmetry of links, the distribution of source node of activity, and temporal evolution of activity propagation. Moreover, we can see from evaluation by the simulation that the conformity between the node centrality and the oscillation energy depends on the value of γ. Finally, in this research, we succeeded in understanding the different centralities (degree centrality and betweenness centrality) in the same framework. This contributes to the basic research field of the social network by understanding the concept of node centrality at a more fundamental level.

Acknowledgments

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Chisa Takano received a B.E. degree in Telecommunication Engineering from Osaka University, Japan, in 2000, and a Ph.D. in System Design Engineering from Tokyo Metropolitan University, Japan, in 2008. In 2000, she joined the Traffic Research Center, NTT Advanced Technology Corporation. Since April 2008, she has been an Associate Professor at the Graduate School of Information Sciences, Hiroshima City University. Her research interests are computer networks and distributed systems. She received the IEICE Young Researchers' Award in 2003. She received the Information Network Research Award from the IEICE in 2004, 2012, and 2015. She is a member of IEEE (U.S.A.) and IPSJ (Japan).

Masaki Aida received his B.S. degree in Physics and M.S. degree in Atomic Physics from St. Paul's University, Tokyo, Japan, in 1987 and 1989, respectively, and his Ph.D. in Telecommunications Engineering from the University of Tokyo, Japan, in 1999. In April 1989, he joined NTT Laboratories. From April 2005 to March 2007, he was an Associate Professor at the Faculty of SystemsDesign, Tokyo Metropolitan University. He has been a Professor of the Graduate School of Systems Design, Tokyo Metropolitan University since April 2007. His current interests include analysis of social network dynamics and distributed control of computer communication networks. He received the Best Tutorial Paper Award and the Best Paper Award of IEICE Communications Society in 2013 and 2016, respectively. He is a member of the IEEE and the Operations Research Society of Japan.