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Dynamical Model of Flaming Phenomena in On-Line Social Networks

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***Abstract*—This paper proposes an oscillation model for describing the *flaming* phenomena in on-line social networks, and discusses countermeasures to flaming based on the proposed oscillation model. The most significant feature of the proposed model is that the cause of flaming can be explained by the structure of the network. Based on the proposed model, we can suppress the generation of flaming by controlling link weights of a part of the network. This passive solution to flaming is important for the stable operation of social media networks.**

I. Introduction

Since information exchange on social networks is being activated by the recent rapid adoption of information networks, the complex dynamics thought to underlie the propagation of activities in on-line social networks have become interesting research subjects [1], [2], [3]. In particular, the explosive phenomenon called *flaming* seriously impacts not only on the stability of information networks but also social activities in the real world.

The spectral graph theory is a powerful approach to the analysis of network dynamics, and is applicable to many problems such as clustering of networks, graph drawing, graph cut, node coloring, and image segmentation [4], [5], [6]. Spectral graph theory is also important for describing diffusion on networks and the Markov process, and the consensus problem [7], [8]. Oscillation dynamics are quite different from the above topics. Recently, we introduced an oscillation model on networks and investigated its properties [9], [10], [11], [12], [13]. The significance of our oscillation model is that the oscillation energy of each node gives a generalized node centrality that describes the strength of the node's activity. Generalized node centrality can reproduce the conventional node centrality (degree centrality and betweenness centrality [14], [15], [16]) in simple cases. As an application of the oscillation model, the network resonance method that can determine the link weight of social networks (the strength of interaction between users) has been investigated [17], [18].

In this paper, we define the flaming phenomenon as divergence of generalized node centrality. We are able to model flaming by generalizing the oscillation model for a general directed graph. In this model, the strength of node activity increases exponentially with time under a certain condition. The most significant feature of the proposed model is that the cause of flaming can be explained by network structure. Based on this feature, we also discuss countermeasures to flaming.

The rest of this paper is organized as follows: In Sec. II, we discuss ways of decomposing directed graphs to *symmetrizable directed graphs* and *one-way link graphs*, and introduce the scaled Laplacian matrix [9], [10]. In addition, we introduce an oscillation model on a symmetrizable directed graph, and the concept of generalized node centrality as the oscillation energy of each node. In Sec III, we propose a generalization of the oscillation model for general directed graphs, and discuss the solution of the equation of motion. In Sec IV, after defining flaming, we propose a model that describes flaming. In addition, we discuss a countermeasure.

II. Preliminary

A. Decomposition of Directed Networks and Symmetrizable Directed Networks

Network structure is frequently expressed as a matrix. Let us consider loop-free directed graph $\mathcal{G}(V, E)$ with n nodes, where $V = \{1, 2, \ldots, n\}$ is the set of nodes and E is the set of directed links. In addition, let the link weight for link $(i \to j) \in E$ be $w_{ij} > 0$. We define the following $n \times n$ square matrix $\boldsymbol{\mathcal{A}} = [\mathcal{A}_{ij}]$ as

$$\mathcal{A}_{ij} := \begin{cases} w_{ij} & ((i \to j) \in E), \\ 0 & ((i \to j) \notin E). \end{cases} \tag{1}$$

This matrix represents link presence and weight, and is called the (weighted) adjacency matrix. Next, we define the weighted out-degree, d_i, of node i $(i = 1, 2, \ldots, n)$ as

$$d_i := \sum_{j \in \partial i} w_{ij}, \tag{2}$$

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where ∂i denotes the set of nodes adjacent to node i. Also, weighted out-degree matrix $\boldsymbol{\mathcal{D}}$ is defined as

$$\boldsymbol{\mathcal{D}} := \mathrm{diag}(d_1, \ldots, d_n).$$

The Laplacian matrix $\boldsymbol{\mathcal{L}}$ of directed graph $\mathcal{G}(V, E)$ is defined as follows [4], [5] (Fig. 1).

$$\boldsymbol{\mathcal{L}} := \boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{A}}, \tag{3}$$

where this is, in general, an asymmetric matrix. Figure 1 shows an example of the Laplacian matrix for directed graphs. Note that since the row sum of the Laplacian matrix is 0, the Laplacian matrix has 0 as an eigenvalue. In addition, it is known that the multiplicity of eigenvalue 0 is equal to the number of connected components of the corresponding graph [4]. So, for a connected graph, the multiplicity is 1. Hereafter, we consider a connected graph without loss of generality.

We now consider how to decompose the Laplacian matrix of a directed graph into two parts: one is the Laplacian matrix for a *symmetrizable* graph, $\boldsymbol{\mathcal{L}}_0$, (the definition is stated below) and the other is the Laplacian matrix for a one-way link graph, $\boldsymbol{\mathcal{L}}_{\mathrm{I}}$, this has at most only one-way directed links between nodes, that is,

$$\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_{\mathrm{I}}. \tag{4}$$

Here, the definition of the symmetrizable graph is as follows. We consider the left eigenvector ${}^t\boldsymbol{m} = (m_1, \ldots, m_n)$ of Laplacian matrix $\boldsymbol{\mathcal{L}}$ associated with eigenvalue 0, that is,

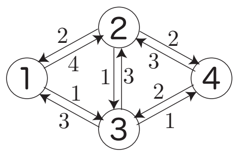
$${}^t\boldsymbol{m}\,\boldsymbol{\mathcal{L}} = {}^t\mathbf{0}, \tag{5}$$

where ${}^t\mathbf{0} := (0, \ldots, 0)$. Note that the multiplicity of eigenvalue 0 is 1 for a connected graph. The directed graph described by $\boldsymbol{\mathcal{L}}$ is symmetrizable iff each component $m_i > 0$ of the left eigenvector ${}^t\boldsymbol{m}$ satisfies

$$m_i\, w_{ij} = m_j\, w_{ji}. \tag{6}$$

The physical meaning of this condition is discussed in Sect. II-C. We denote the Laplacian matrix for a symmetrizable graph as $\boldsymbol{\mathcal{L}}_0$.

Let $G(V, E)$ be an undirected graph whose link weight between node i–j is $k_{ij} := m_i\, w_{ij}$. Since $k_{ij} = k_{ji}$ from



$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} 0 & 4 & 1 & 0 \\ 2 & 0 & 1 & 2 \\ 3 & 3 & 0 & 1 \\ 0 & 3 & 2 & 0 \end{bmatrix}$$

$$\boldsymbol{\mathcal{D}} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \boldsymbol{\mathcal{L}} = \begin{bmatrix} 5 & -4 & -1 & 0 \\ -2 & 5 & -1 & -2 \\ -3 & -3 & 7 & -1 \\ 0 & -3 & -2 & 5 \end{bmatrix}$$

Fig. 1: Example of Laplacian matrix $\boldsymbol{\mathcal{L}}$

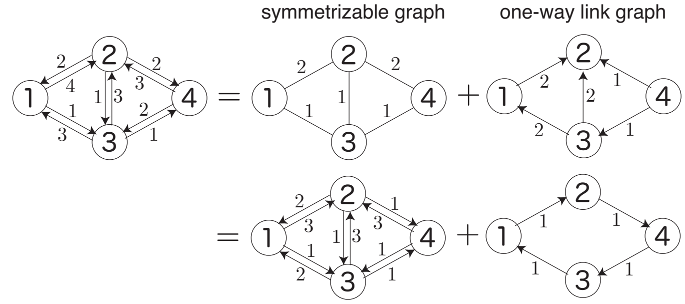
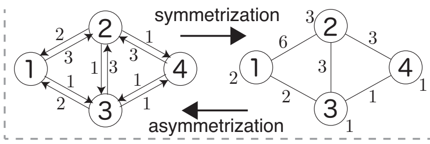


Fig. 2: Examples of decomposition $\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_{\mathrm{I}}$ (1)

$$\boldsymbol{\mathcal{L}}_0 = \begin{bmatrix} 4 & -3 & -1 & 0 \\ -2 & 4 & -1 & -1 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$



$$= \begin{bmatrix} 4 & -6/2 & -2/2 & 0 \\ -6/3 & 4 & -3/3 & -3/3 \\ -2/1 & -3/1 & 6 & -1/1 \\ 0 & -3/1 & -1/1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 & -2 & 0 \\ -6 & 12 & -3 & -3 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$

Fig. 3: Symmetrization of a symmetrizable Laplacian matrix $\boldsymbol{\mathcal{L}}_0$

(6), the corresponding Laplacian matrix $\boldsymbol{L}$ is symmetric. By using $\boldsymbol{L}$, the symmetrizable Laplacian matrix $\boldsymbol{\mathcal{L}}_0$ is expressed as

$$\boldsymbol{\mathcal{L}}_0 = \boldsymbol{M}^{-1}\,\boldsymbol{L}, \tag{7}$$

where $\boldsymbol{M} := \mathrm{diag}(m_1, \ldots, m_n)$ means the scaling factors of nodes. An example of the symmetrizable Laplacian matrix is shown later with a discussion of its decomposition.

Figure 2 shows examples of decomposition (4) of the directed graph depicted in Fig. 1. The first decomposition of Fig. 2 yields an undirected graph and a one-way link graph. This decomposition is uniquely determined by choosing the components of $\boldsymbol{\mathcal{L}}_0 = [\ell^0_{ij}]$ and $\boldsymbol{\mathcal{L}}_{\mathrm{I}} = [\ell^{\mathrm{I}}_{ij}]$ as

$$\ell^0_{ij} = -\min(w_{ij}, w_{ji}), \tag{8}$$

$$\ell^{\mathrm{I}}_{ij} = -w_{ij} + \min(w_{ij}, w_{ji}), \tag{9}$$

for $i \neq j$. The second decomposition of Fig. 2 yields a symmetrizable directed graph and an one-way link graph. The Laplacian matrix of the symmetrizable graph in the second decomposition is expressed in the form of (7) as shown in Fig. 3. The undirected graph is a special case of symmetrizable graphs and $\boldsymbol{M} = \boldsymbol{I}$ (unit matrix) for undirected graphs.

From the above discussion, we can recognize the following facts:

- Any directed graph can be decomposed into a symmetrizable directed graph and a one-way link graph.
- The decomposition is not unique, as it depends on the choice of symmetrizable directed graph included in the original directed graph.

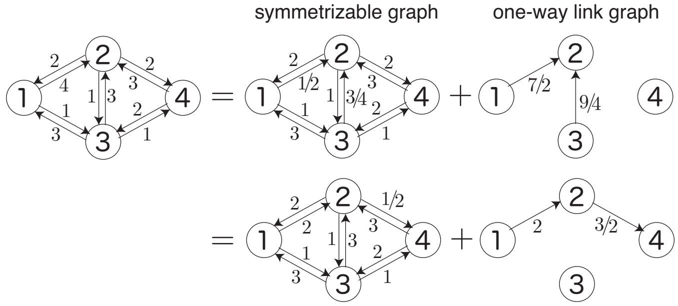


Fig. 4: Examples of decomposition $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathrm{I}$ (2)

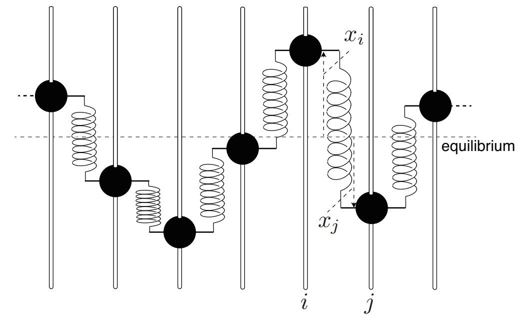


Fig. 5: Oscillation model on networks.

If the original directed graph is symmetrizable, we can choose a decomposition such that $\mathcal{L}_\mathrm{I} = \mathbf{O}$ (the null matrix).

Figure 4 shows other examples of decomposition (4) of the directed graph depicted in Fig. 1. As discussed later, the degree of freedom of the decomposition (the degree of freedom of choice in symmetrizable directed graph) plays a significant role in controlling network dynamics.

B. Scaled Laplacian Matrix and Its Decomposition

First, we define the scaled Laplacian matrix for a symmetrizable network as

$$\boldsymbol{S}_0 := \boldsymbol{M}^{+1/2}\,\mathcal{L}_0\,\boldsymbol{M}^{-1/2} = \boldsymbol{M}^{-1/2}\,\boldsymbol{L}\,\boldsymbol{M}^{-1/2}, \qquad (10)$$

where $\boldsymbol{M} := \mathrm{diag}(m_1, \ldots, m_n)$ is the scaling factors for symmetrizable graph $\mathcal{L}_0$. From the second equality of (10), $\boldsymbol{S}_0$ is a symmetric matrix.

Next, we introduce the scaled Laplacian matrix, $\boldsymbol{S}$, for general directed graphs. The most significant characteristic of $\boldsymbol{S}$ is that the definition of $\boldsymbol{S}$ depends on the choice of symmetrizable directed graph made in the decomposition of (4). Starting with $\mathcal{L}_0$ and its scaling factor $\boldsymbol{M}$, we define the scaled Laplacian matrix $\boldsymbol{S}$ for a general directed graph as

$$\boldsymbol{S} := \boldsymbol{M}^{+1/2}\,\mathcal{L}\,\boldsymbol{M}^{-1/2}. \qquad (11)$$

$\boldsymbol{S}$ is decomposed as

$$\boldsymbol{S} = \boldsymbol{S}_0 + \boldsymbol{S}_\mathrm{I}, \qquad (12)$$

where

$$\boldsymbol{S}_\mathrm{I} := \boldsymbol{M}^{+1/2}\,\mathcal{L}_\mathrm{I}\,\boldsymbol{M}^{-1/2}.$$

It is noteworthy that the definition of $\boldsymbol{S}$ depends on the choice of $\mathcal{L}_0$, and decomposition (12) also depends on the choice of $\mathcal{L}_0$.

Here, we summarize two important characteristics with respect to the scaled Laplacian matrix.

Let us consider the eigenvalue equation of $\mathcal{L}$ as

$$\mathcal{L}\,\boldsymbol{x} = \bar{\lambda}\,\boldsymbol{x},$$

where $\bar{\lambda}$ is an eigenvalue of $\mathcal{L}$ and $\boldsymbol{x}$ is the eigenvector associated with the eigenvalue $\bar{\lambda}$. By multiplying $\boldsymbol{M}^{+1/2}$ from the left, we have

$$\boldsymbol{M}^{+1/2}\,\mathcal{L}\,\boldsymbol{x} = \boldsymbol{S}\,\boldsymbol{M}^{+1/2}\,\boldsymbol{x} = \bar{\lambda}\,\boldsymbol{M}^{+1/2}\,\boldsymbol{x}.$$

This means scaled Laplacian matrix $\boldsymbol{S}$ has the same eigenvalues as $\mathcal{L}$, and its eigenvector is $\boldsymbol{y} := \boldsymbol{M}^{+1/2}\,\boldsymbol{x}$. Thus, the characteristics of a directed graph expressed by $\mathcal{L}$ can be investigated through $\boldsymbol{S}$.

Next, we consider $\boldsymbol{S}_0$ of decomposition (12). Since $\boldsymbol{S}_0$ is a semi-positive definite matrix, let us consider sorting the eigenvalues of $\boldsymbol{S}_0$ (also $\mathcal{L}_0$) in ascending order,

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1}. \qquad (13)$$

In addition, since $\boldsymbol{S}_0$ is a real symmetric matrix, we can choose eigenvector $\boldsymbol{v}_\mu$ $(\mu = 0, 1, \ldots, n-1)$ as the orthonormal eigenvector associated with λ_μ. That is,

$$\boldsymbol{S}_0\,\boldsymbol{v}_\mu = \lambda_\mu\,\boldsymbol{v}_\mu, \quad \boldsymbol{v}_\mu \cdot \boldsymbol{v}_\nu = \delta_{\mu\nu}, \qquad (14)$$

where $\delta_{\mu\nu}$ denotes the Kronecker delta. Selecting $\mathcal{L}_0$ (or, equivalently, defining $\boldsymbol{S}$) corresponds to selecting a basis of the n-dimensional state space. Thus, we can choose an appropriate basis of the n-dimensional state space based on the degree of freedom of the decomposition of (4).

C. Oscillation Model on a Symmetrizable Directed Graph

Let weight x_i of node i be displacement from equilibrium, and let its restoring force be proportional to the difference in the displacements of adjacent nodes. Figure 5 is a representative image of our oscillation model. Although the figure shows a 1-dimensional network, it is easily extended to general networks. To represent diverse oscillating behaviors, we allow the spring constant of each link to be different and each node to have different mass.

Here, it is worthwhile discussing the validity of the oscillation model given that its restoring force is proportional to the difference in displacements. Let the restoring force of node i from node j be a function $f(\Delta x)$ of the difference $\Delta x := x_i - x_j$ of the displacements of adjacent nodes i and j. It is natural to assume $f(\Delta x) = 0$ if $\Delta x = 0$. For small Δx, we can expand $f(\Delta x)$ as

$$f(\Delta x) = -k_{ij}\,\Delta x + O(\Delta x^2), \qquad (15)$$

where k_{ij} is a positive constant corresponding to the spring constant. Our oscillation model can be considered as a basic and universal model if the nonlinear effects in $O(\Delta x^2)$ are relatively small.

We assign a spring constant to each link and express it as link weight $k_{ij} > 0$. In addition, we assign mass $m_i > 0$ to node i. The equation of motion of the node displacement vector $\boldsymbol{x}(t) := {}^t(x_1(t), \ldots, x_n(t))$ for non-damped oscillation on networks is obtained as follows (see Appendix):

$$\boldsymbol{M}\,\frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2} = -\boldsymbol{L}\,\boldsymbol{x}(t),$$

or, by multiplying $\boldsymbol{M}^{-1}$ from the left,

$$\frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2} = -\boldsymbol{\mathcal{L}}_0\,\boldsymbol{x}(t), \tag{16}$$

where $\boldsymbol{M}$ is the mass matrix $\boldsymbol{M} := \mathrm{diag}(m_1, \ldots, m_n)$. Note that the equation of motion (16) reflects the asymmetric characteristics of links described by the asymmetric Laplacian matrix (7). Also note that $\boldsymbol{\mathcal{L}}_0$ is a Laplacian matrix for the symmetrizable directed graph. Node characteristic m_i in (6) corresponds to the mass of node i in the oscillation model. In addition, condition (6) represents Newton's third law (the law of action and reaction). To symmetrize the equation of motion, we introduce vector $\boldsymbol{y}$ which is defined by

$$\boldsymbol{y}(t) = \boldsymbol{M}^{+1/2}\,\boldsymbol{x}(t),$$

where the equation of motion is written as

$$\frac{\mathrm{d}^2\boldsymbol{y}(t)}{\mathrm{d}t^2} = -\boldsymbol{S}_0\,\boldsymbol{y}(t). \tag{17}$$

It follows that the equation of motion will yield the eigenvalue problem of the symmetric scaled Laplacian matrix.

Let $\boldsymbol{y}(t)$ be expanded by the eigenbasis $\boldsymbol{v}_\mu$ of $\boldsymbol{S}_0$ as

$$\boldsymbol{y}(t) = \sum_{\mu=0}^{n-1} a_\mu(t)\,\boldsymbol{v}_\mu.$$

Substituting it into the equation of motion (17) yields the equations of motion for the Fourier mode $a_\mu(t)$ ($\mu = 0, 1, \ldots, n-1$) as follows,

$$\frac{\mathrm{d}^2 a_\mu(t)}{\mathrm{d}t^2} = -\lambda_\mu\,a_\mu(t). \tag{18}$$

The solution of (18) is given by

$$a_\mu(t) = a_\mu(0)\,e^{\pm\mathrm{i}\,\omega_\mu\,t}, \tag{19}$$

where $\omega_\mu = \sqrt{\lambda_\mu}$, $\mathrm{i} = \sqrt{-1}$. The initial condition, $a_\mu(0) = |a_\mu(0)|\,e^{\mathrm{i}\theta_\mu}$ $(-\pi < \theta_\mu \le \pi)$, gives the amplitude $|a_\mu(0)|$ and phase θ_μ of the corresponding Fourier mode, μ.

The solution (19) means that the oscillation dynamics on symmetrizable directed networks can be expressed by using decomposition into the equation of motion of the harmonic oscillator (18) for each Fourier mode. The solution of oscillation on networks (the solution of (16)) is expressed as

$$\boldsymbol{x}(t) = \boldsymbol{M}^{-1/2}\left(\sum_{\mu=0}^{n-1} a_\mu(0)\,e^{\pm\mathrm{i}\,\omega_\mu\,t}\,\boldsymbol{v}_\mu\right). \tag{20}$$

In closed dynamic models, any oscillation is damped over time due to frictional force. Let us consider the equation of motion for damped oscillation

$$\boldsymbol{M}\,\frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2} + \gamma\,\boldsymbol{M}\,\frac{\mathrm{d}\,\boldsymbol{x}(t)}{\mathrm{d}t} = -\boldsymbol{L}\,\boldsymbol{x}(t), \tag{21}$$

where γ is a constant. Here $\gamma\boldsymbol{M}$ means the viscous damping coefficient, where it is important to note that the viscous damping coefficient is assumed to be proportional to node mass. By using vector $\boldsymbol{y}(t) = \boldsymbol{M}^{+1/2}\,\boldsymbol{x}(t)$, we can symmetrize the equation of motion as

$$\frac{\mathrm{d}^2\boldsymbol{y}(t)}{\mathrm{d}t^2} + \gamma\,\frac{\mathrm{d}\,\boldsymbol{y}(t)}{\mathrm{d}t} = -\boldsymbol{S}_0\,\boldsymbol{y}(t). \tag{22}$$

The equation of motion for Fourier mode $a_\mu(t)$ is expressed as

$$\frac{\mathrm{d}^2 a_\mu(t)}{\mathrm{d}t^2} + \gamma\,\frac{\mathrm{d}\,a_\mu(t)}{\mathrm{d}t} + \lambda_\mu\,a_\mu(t) = 0. \tag{23}$$

To analyze the solution of this equation, we assume the solution takes the form of $a_\mu(t) \propto e^{\alpha t}$. By substituting this into the equation of motion, we obtain the characteristic equation

$$\alpha^2 + \gamma\alpha + \lambda_\mu = 0. \tag{24}$$

The form of solution to the equation of motion is different depending on the solution of the characteristic equation, $\alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \lambda_\mu}$. In the case of $(\gamma/2)^2 < \lambda_\mu$, the solution describes damped oscillations,

$$a_\mu(t) = a_\mu(0)\,\exp\left[-\frac{\gamma}{2}\,t \pm \mathrm{i}\sqrt{\lambda_\mu - (\gamma/2)^2}\,t\right]. \tag{25}$$

On the other hand, if $(\gamma/2)^2 \ge \lambda_\mu$, the solution is not oscillating because the strength of damping is too great.

D. Generalized Node Centrality

The importance of the oscillation model lies in the relationship between the oscillation energy of each node and node centrality.

From (19), oscillation behavior is decomposed into n independent harmonic oscillators. Since the oscillation energy of a harmonic oscillator with mass m, natural frequency ω, and amplitude A is given by $\frac{1}{2}\,m\,\omega^2 A^2$, the oscillation energy E_i of node i is obtained by summing the oscillation energy for each Fourier mode,

$$\begin{aligned} E_i &= \frac{1}{2}\,m_i \sum_{\mu=0}^{n-1} \left|\,\omega_\mu\,a_\mu(t)\,\frac{v_\mu(i)}{\sqrt{m_i}}\,\right|^2 \\ &= \frac{1}{2}\sum_{\mu=0}^{n-1} |\,\omega_\mu\,a_\mu(t)\,v_\mu(i)\,|^2, \end{aligned} \tag{26}$$

where $v_\mu(i)$ denotes the i-th component of the eigenbasis $\boldsymbol{v}_\mu$ associated with the eigenvalue λ_μ of the scaled Laplacian matrix $\boldsymbol{S}_0$, that is,

$$\boldsymbol{v}_\mu = {}^t(v_\mu(1), \ldots, v_\mu(n)).$$

As the initial condition of the wave equation (16), we use node displacement. We call this node the source node of activity. Let us consider the situation that the source node of activity is chosen at random. If we set all link weights (node masses are set at 1), the oscillation energy of each node gives the degree centrality. On the other hand, we consider the shortest paths between all pairs of nodes. If we set all link weights as the number of the shortest paths passing through the link, the oscillation energy of each node gives the betweenness centrality. Assigning link weights and node masses flexibly yields a generalized form of node centrality. The oscillation energy can be defined even for damped oscillation on networks. A detailed discussion is presented in [11], [12].

For damped oscillation, the oscillation energy depends on time. From (25) and (26), the time-dependent oscillation energy $E_i(t)$ for node i is given by

$$E_i(t) = E_i \, e^{-\gamma t}, \tag{27}$$

where $E_i = E_i(0)$. For non-damped oscillation $\gamma = 0$, $E_i(t)$ is time independent

$$E_i(t) = E_i.$$

The total oscillation energy of the network is obtained as the sum of the oscillation energy of each node,

$$\begin{aligned} E(t) &= \sum_{i=1}^{n} E_i(t) \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{\mu=0}^{n-1} |\, \omega_\mu \, a_\mu(t) \, v_\mu(i) \,|^2 \\ &= \frac{1}{2} \sum_{\mu=0}^{n-1} |\, \omega_\mu \, a_\mu(t) \,|^2 \\ &= \frac{1}{2} \, |\boldsymbol{z}(t)|^2 , \end{aligned} \tag{28}$$

where

$$\boldsymbol{z}(t) = \sum_{\mu=0}^{n-1} \omega_\mu \, a_\mu(t) \, \boldsymbol{v}_\mu .$$

III. Generalized Oscillation Model on General Directed Networks

Here, we extend the oscillation model described in Sect. II-C to a general directed graph.

Let $x_i(t)$ be the displacement of node i at time t on directed graph $\mathcal{G}(V, E)$, and the restoring force that node i receives from adjacent node j be $-w_{ij}\,(x_i(t) - x_j(t))$. Note that since $w_{ij} \neq w_{ji}$, the strength of restoring forces between nodes i–j are asymmetric, in general.

Then, the equation of motion of the displacement vector $\boldsymbol{x}(t) = {}^t(x_1(t), \ldots, x_n(t))$ can be written as

$$\frac{\mathrm{d}^2 \boldsymbol{x}(t)}{\mathrm{d}t^2} = -\boldsymbol{\mathcal{L}} \, \boldsymbol{x}(t) = -(\boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_\mathrm{I}) \, \boldsymbol{x}(t), \tag{29}$$

and the equation of motion of vector $\boldsymbol{y}(t) = \boldsymbol{M}_0^{+1/2} \, \boldsymbol{x}(t)$ is obtained as

$$\frac{\mathrm{d}^2 \boldsymbol{y}(t)}{\mathrm{d}t^2} = -\boldsymbol{S} \, \boldsymbol{y}(t) = -(\boldsymbol{S}_0 + \boldsymbol{S}_\mathrm{I}) \, \boldsymbol{y}(t). \tag{30}$$

Here, it is worthwhile noting that if the oscillation occurs on an unsymmetrizable graph i.e. $\boldsymbol{\mathcal{L}}_\mathrm{I} \neq \mathbf{O}$, Newton's third law does not hold. This means that the generalized oscillation model cannot be depicted by a dynamical model like Fig. 5. In other words, the generalized oscillation model does not correspond to dynamical phenomena in the real world, but to a kind of virtual phenomena in cyber space.

Similar to (22), the equation of motion of damped oscillation can be written as

$$\frac{\mathrm{d}^2 \boldsymbol{y}(t)}{\mathrm{d}t^2} - \gamma \, \frac{\mathrm{d} \, \boldsymbol{y}(t)}{\mathrm{d}t} = -\boldsymbol{S} \, \boldsymbol{y}(t) = -(\boldsymbol{S}_0 + \boldsymbol{S}_\mathrm{I}) \, \boldsymbol{y}(t), \tag{31}$$

where $\gamma \geq 0$ (it includes (30) for $\gamma = 0$).

Since $\boldsymbol{S}$ is no longer a real symmetric matrix, we should consider the following issues when solving the equation of motion (31):

- $\boldsymbol{S}$ is not always diagonalizable.
- Eigenvectors of $\boldsymbol{S}$ cannot always be orthogonalized.
- Eigenvalues of $\boldsymbol{S}$ are, in general, complex numbers.

Let us consider the impact of these issues on the solution of equation of motion (31) and the oscillation energy.

Let the eigenvalues of $\boldsymbol{S}$ be $\bar{\lambda}_\mu$ $(\mu = 0, \ldots, n-1)$ and the eigenvector associated with $\bar{\lambda}_\mu$ be $\bar{\boldsymbol{v}}_\mu$. The necessary condition that $\boldsymbol{S}$ is not diagonalizable is that $\boldsymbol{S}$ has multiple eigenvalues. In other words, the eigenvalue equation

$$\det(\boldsymbol{S} - \bar{\lambda} \, \boldsymbol{I}) = 0, \tag{32}$$

has repeated roots, where $\boldsymbol{I}$ is the unit matrix. In engineering, the value of link weights of actual networks is determined to the accuracy level of the significant digits expected. Therefore, the mathematical condition for repeated roots of (32) is easily avoided by changing the link weights only very slightly. This allows us to assume that $\boldsymbol{S}$ has n distinct eigenvalues.

The fact that the eigenvectors are not orthogonal means that the total oscillation energy is not given by the simple summation of oscillation energy of each oscillation mode, and thus different oscillation modes are coupled. This complicates the expression of the oscillation energy of the network.

The eigenvectors associated with the different eigenvalues are generally not orthogonal, but it is guaranteed that they are linearly independent. Therefore, by using the following $n \times n$ square matrices

$$\bar{\boldsymbol{\Lambda}} := \mathrm{diag}(\bar{\lambda}_0, \ldots, \bar{\lambda}_{n-1}), \text{ and } \bar{\boldsymbol{P}} := (\bar{\boldsymbol{v}}_0, \ldots, \bar{\boldsymbol{v}}_{n-1}),$$

$\boldsymbol{S}$ can be diagonalized as follows:

$$\bar{\boldsymbol{\Lambda}} = \bar{\boldsymbol{P}}^{-1} \, \boldsymbol{S} \, \bar{\boldsymbol{P}}, \tag{33}$$

where the existence of $\bar{\boldsymbol{P}}^{-1}$ arises from the linear independence of $\bar{\boldsymbol{v}}_\mu$. Incidentally, the eigenvalues of $\boldsymbol{S}_0$ are non-negative and all the components of the eigenvectors have real values. However, the eigenvalues of $\boldsymbol{S}$ are generally complex numbers, and the components of the eigenvectors are also generally complex numbers.

Let us expand the solution $\boldsymbol{y}(t)$ of (31) by the eigenvectors of $\boldsymbol{S}$, as

$$\boldsymbol{y}(t) = \sum_{\mu=0}^{n-1} \bar{a}_\mu(t)\, \bar{\boldsymbol{v}}_\mu.$$

By substituting this into the equation of motion (31), and applying a procedure similar to that described in II-C, the characteristic function for Ansatz $\bar{a}_\mu(t) \propto e^{\bar{\alpha} t}$ is obtained as

$$\bar{\alpha}^2 + \gamma\, \bar{\alpha} + \bar{\lambda}_\mu = 0.$$

Since $\bar{\lambda}_\mu$ is a complex number, we define

$$r\, e^{\mathrm{i}\,\theta} := \bar{\lambda}_\mu - (\gamma/2)^2 = -(\bar{\alpha} + \gamma/2)^2,$$

where $r \geq 0$ and $-\pi < \theta \leq \pi$. Then, the solution of the characteristic function is obtained as

$$\bar{\alpha} = -\frac{\gamma}{2} \pm \mathrm{i}\sqrt{r}\, e^{\mathrm{i}\theta/2}.$$

This yields the oscillating solution of $a_\mu(t)$ as

$$\bar{a}_\mu(t) = \bar{a}_\mu(0)\, \exp\left[-\frac{\gamma}{2}\, t \pm \mathrm{i}\, \left(\sqrt{r}\, e^{\mathrm{i}\theta/2}\right)\, t\right]. \qquad (34)$$

IV. Flaming Phenomenon in On-Line Social Networks

This section proposes a dynamical model of the *flaming* phenomenon in on-line social networks, and a countermeasure.

A. Definition of Flaming in Networks

The term *flaming* is a phenomenon arising in on-line social network that strongly impacts not only the on-line social networks themselves but also on off-line social networks. Although flaming causes various social impacts, a common feature of these explosive phenomena is it is caused by the behavior of users.

In our oscillation model, the strength of activity of a network can be expressed by its oscillation energy. Thus, we introduce the total oscillation energy $E(t)$ of the network as

$$E(t) := \frac{1}{2}\, |\bar{\boldsymbol{z}}(t)|^2 = \frac{1}{2} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} \sqrt{|\bar{\lambda}_\mu|\, |\bar{\lambda}_\nu|}\, |\bar{a}_\mu(t)\, \bar{a}_\nu(t)|^2\, (\bar{\boldsymbol{v}}_\mu \cdot \bar{\boldsymbol{v}}_\nu), \qquad (35)$$

where

$$\bar{\boldsymbol{z}}(t) = \sum_{\mu=0}^{n-1} \sqrt{|\bar{\lambda}_\mu|}\, \bar{a}_\mu(t)\, \bar{\boldsymbol{v}}_\mu.$$

Different from the case of symmetrizable graphs, the oscillation energy cannot be divided into the oscillation energy for each oscillation mode. This is because the eigenvectors $\bar{\boldsymbol{v}}_\mu$'s are in general, not orthogonal, and this causes coupling between different oscillation modes. Note that if the network is symmetrizable, $\boldsymbol{S} = \boldsymbol{S}_0$, the eigenvalue $\bar{\lambda}_\mu$ is real and the eigenvectors are orthogonal. Then, (35) is reduced to (28).

Here, we define flaming in on-line social networks. We expect that the strength of explosive behavior of users can be measured by oscillation energy. That is, flaming is defined as the divergence of oscillation energy,

$$\lim_{t\to\infty} E(t) = \infty. \qquad (36)$$

Since the time dependent factors of $E(t)$ are present only in $\bar{a}_\mu(t)$'s, we need focus only on the behavior of $\bar{a}_\mu(t)$'s. Therefore, flaming occurs if

$$\lim_{t\to\infty} |\bar{a}_\mu(t)| = \infty, \qquad (37)$$

for some μ ($\mu = 0, 1, \ldots, n-1$).

B. Modeling of Flaming in Network Dynamics

In order to model flaming, let us consider the behavior of (34). Since

$$\sqrt{r}\, e^{\mathrm{i}\theta/2} = \sqrt{r}\, \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\, \sqrt{r}\, \sin\left(\frac{\theta}{2}\right), \qquad (38)$$

we have

$$\bar{a}_\mu(t) = \bar{a}_\mu(0)\, \exp\left[-\left(\frac{\gamma}{2} \pm \sqrt{r}\, \sin\left(\frac{\theta}{2}\right)\right)\, t \pm \mathrm{i}\, \sqrt{r}\, \cos\left(\frac{\theta}{2}\right)\, t\right]. \qquad (39)$$

This means that the imaginary part of the eigenvalue $\bar{\lambda}_\mu$ contributes to the amplitude of the oscillation mode. From the structure of (39), we can recognize that $\bar{a}_\mu(t)$ diverges exponentially if

$$\sqrt{r}\, \left|\sin\left(\frac{\theta}{2}\right)\right| > \frac{\gamma}{2}.$$

This mechanism can be interpreted as the cause of the flaming phenomenon in on-line social networks. Examples of the behavior of $\bar{a}_\mu(t)$ for $\sqrt{r}\, |\sin(\theta/2)\,| < \gamma/2$ and $\sqrt{r}\, |\sin(\theta/2)\,| > \gamma/2$ are shown in Figs. 6 and 7, respectively.

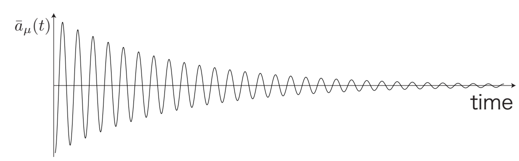


Fig. 6: Example of the behavior of $\bar{a}_\mu(t)$ where $\sqrt{r}\, |\sin(\theta/2)\,| < \gamma/2$.

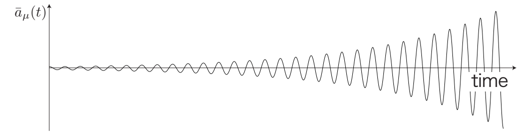


Fig. 7: Example of the behavior of $\bar{a}_\mu(t)$ where $\sqrt{r}\, |\sin(\theta/2)\,| > \gamma/2$.

C. Flaming Countermeasure

In order to prevent and/or suppress flaming, an effective control strategy is to make the network symmetrizable. This approach makes $\boldsymbol{\mathcal{L}}_{\mathrm{I}} = \mathbf{O}$ based on the decomposition (4). At this time, the diversity of decomposition (4) leads to control flexibility.

Regardless of the value of $\gamma \geq 0$, in order to always prevent the outbreak of flaming, it suffices if all the eigenvalues of $\boldsymbol{S}$ are real numbers. The condition that the directed graph $\mathcal{G}(V, E)$ is symmetrizable is a sufficient condition that all the eigenvalues of $\boldsymbol{S}$ are real numbers. Therefore, we can prevent flaming by making $\mathcal{G}(V, E)$ symmetrizable.

Let us consider nodes $i_1, i_2, \ldots, i_k$ (in a loop), and the product of kink weights along $i_1 \to i_2 \to \cdots \to i_k \to i_1$. If all pairs of adjacent nodes satisfy the condition (6), the product of link weights is written as

$$\begin{aligned} w_{i_1 i_2}\, w_{i_2 i_3} \cdots w_{i_k i_1} &= \frac{m_{i_2}}{m_{i_1}}\, w_{i_2 i_1}\, \frac{m_{i_3}}{m_{i_2}}\, w_{i_3 i_2} \cdots \frac{m_{i_1}}{m_{i_k}}\, w_{i_1 i_k} \\ &= w_{i_1 i_k} \cdots w_{i_3 i_2}\, w_{i_2 i_1}. \end{aligned} \tag{40}$$

That is, the product of link weights through the loop in one direction is equivalent to the product in the opposite direction. Conversely, if (40) holds, we can choose m_i and m_j that satisfy (6). Therefore, it is possible to symmetrize a directed graph by adjusting the product of link weights in the clockwise direction so that it equals that in the counterclockwise direction, for all loops.

Next, we consider two loops C_1 and C_2 as shown in the upper figure of Fig. 8, and let the products of link weights in the clockwise and the counterclockwise directions be R_1 and L_1 for loops C_1, and be R_2 and L_2 for C_2, respectively. In addition, let loops C_1 and C_2 share a part of their paths as shown in the lower figure of Fig. 8. For the shared path, let the products of link weights in the upward and downward directions be U and D, respectively. If loops C_1 and C_2 satisfy the condition (6) (i.e. $R_1 = L_1$ and $R_2 = L_2$), the products of link weights of the combined loop C_{1+2} satisfy

$$\frac{R_1}{D}\,\frac{R_2}{U} = \frac{L_1}{U}\,\frac{L_2}{D}. \tag{41}$$

This means the products of link weight in the clockwise and counterclockwise directions of the combined loop C_{1+2} are equal. Therefore, it is sufficient to adjust the link weight for undividable loops.

Note that we can adjust the products of link weights of a loop regardless of which link we select. This fact is related to the fact that decomposition (4) does not have a unique solution.

V. Conclusion

This paper has introduced a generalization of the oscillation model on directed networks by adding the effect of one-way link topology to the conventional oscillation model. The proposed generalized oscillation can describe general asymmetric interactions between nodes. We have shown that the solution of the generalized oscillation model depends on the eigenvalues of the asymmetric scaled Laplacian matrix, and from the fact that the eigenvalues are complex numbers in general, we have clarified a framework that can explain the exponential increase of oscillation energy. This framework can explain why explosive behaviors like flaming can autonomously breakout in networks. Based on the framework, we also discussed countermeasures to flaming.

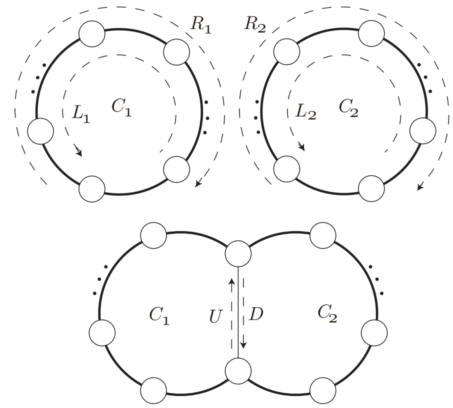


Fig. 8: The product of link weights with respect to the combined closed loops

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